Available online at www.sciencedirect.com
ScienceDirect
Advances in Mathematics 219 (2008) 568–603
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

Hearing the type of a domain in $\mathbb{C}^2$ with the $\bar{\partial}$-Neumann Laplacian

Siqi Fu [1]

Department of Mathematical Sciences, Rutgers University-Camden, Camden, NJ 08102, USA

Received 19 May 2006; accepted 8 May 2008

Available online 5 June 2008

Communicated by Charles Fefferman

Abstract

A smooth bounded pseudoconvex domain in $\mathbb{C}^2$ is of finite type if and only if the number of eigenvalues of its $\bar{\partial}$-Neumann Laplacian that are less than or equal to λ has at most polynomial growth as λ goes to infinity.

Keywords: $\bar{\partial}$-Neumann Laplacian; Pseudoconvexity; Finite type; Spectral kernel; Wavelet

Contents

E-mail address: sfu@camden.rutgers.edu.

[1] The author was supported in part by NSF grants DMS 0500909/0805852 and by an AMS Centennial Research Fellowship.

0001-8708/$ – see front matter
doi:10.1016/j.aim.2008.05.003

1. Introduction

Motivated by Mark Kac's famous question "Can one hear the shape of a drum?" (see [35,31, 6]), we study the interplays between the geometry of a bounded domain in $\mathbb{C}^n$ and the spectrum of the $\bar\partial$-Neumann Laplacian. Since the work of Kohn [36,37], it has been discovered that regularity of the $\bar\partial$-Neumann Laplacian is intimately connected to the boundary geometry. (See, for example, the surveys [5,13,19,28] and the books [24,34,41,11,52].) It is then natural to expect that one can "hear" more about the geometry of a bounded domain in $\mathbb{C}^n$ with the $\bar\partial$-Neumann Laplacian than with the usual Dirichlet and Neumann Laplacians.

For a bounded domain in $\mathbb{C}^n$, it follows from Hörmander's L^2-estimates for the $\bar\partial$-operator [33] that pseudoconvexity implies positivity of the $\bar\partial$-Neumann Laplacian on all $(0, q)$-forms, $1 \leqslant q \leqslant n-1$. The converse is also true (under the assumption that the interior of the closure of the domain is the domain itself). This is a consequence of the sheaf cohomology theory dated back to Oka and H. Cartan—it is proved, implicitly, in [54,43,51]. (See [26] for a discussion of this and other related facts without the sheaf cohomology theory.) Therefore, in Kac's language, we can "hear" pseudoconvexity via the $\bar\partial$-Neumann Laplacian.

Regularity and spectral theories of the $\bar\partial$-Neumann Laplacian closely intertwine. For example, on the one hand, by a classical theorem of Hilbert in operator theory, the $\bar\partial$-Neumann Laplacian has compact resolvent if and only if its essential spectrum is empty. On the other hand, by a classical result of Kohn and Nirenberg [40], for a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, compactness of the $\bar\partial$-Neumann operator (the inverse of the $\bar\partial$-Neumann Laplacian) implies exact global regularity of the $\bar\partial$-Neumann Laplacian on L^2-Sobolev spaces. It was shown in [27] that for a bounded convex domain in $\mathbb{C}^n$, the $\bar\partial$-Neumann operator on $(0, q)$-forms is compact if and only if the boundary contains no q-dimensional complex varieties. However, such characterization does not hold even for a pseudoconvex Hartogs domain in $\mathbb{C}^2$ ([45]; see also [28]). It was observed in [29] that compactness in the $\bar\partial$-Neumann problem on a complete Hartogs domain in $\mathbb{C}^2$ is closely related to diamagnetism and paramagnetism for certain Schrödinger operators with infinitely degenerated magnetic fields. The desired paramagnetic property in semi-classical limits was recently established in [15]. As a consequence, for a smooth bounded pseudoconvex Hartogs domain in $\mathbb{C}^2$, compactness of the $\bar\partial$-Neumann operator on $(0, 1)$-forms implies that the boundary contains no pluripotentials (more precisely, it satisfies property (P) in the sense of Catlin [9] or equivalently is B-regular in the sense of Sibony [55]). This, together with an earlier result of Catlin [9] (compare [56]), shows that one can determine whether or not the boundary of a Hartogs domain in $\mathbb{C}^2$ contains pluripotentials via the spectrum of the $\bar\partial$-Neumann Laplacian.

In this paper, we continue our study of spectral theory of the $\bar\partial$-Neumann Laplacian. Our main result can be stated as follows.

Theorem 1.1. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^2$. Let $\mathcal{N}(\lambda)$ be the number of eigenvalues of the $\bar\partial$-Neumann Laplacian that are less than or equal to λ. Then $b\Omega$ is of finite type if and only if $\mathcal{N}(\lambda)$ has at most polynomial growth.*

Recall that a smooth bounded domain in $\mathbb{C}^n$ is strictly pseudoconvex if for each boundary point, there is a local change of holomorphic coordinates which makes the boundary near this point strictly convex (in the sense that the real Hessian of the defining function is positively definite on the real tangent space). A domain in $\mathbb{C}^n$ is pseudoconvex if it can be exhausted by strictly pseudoconvex domains. (We refer the reader to the textbook [42] for an exposition of these concepts.) For a smooth domain Ω in $\mathbb{C}^2$, the type of its boundary $b\Omega$ (in the sense of

Kohn [38]) is the maximal order of contact of a regular complex variety with $b\Omega$. (See [4,39, 16,8,18] for in depth treatises of related notions of finite type in $\mathbb{C}^n$. All the notions coincide in $\mathbb{C}^2$.)

Theorem 1.1 consists of two parts. More precisely, for the sufficiency, we establish the following result.

Theorem 1.2. *Let $\Omega \subset \mathbb{C}^2$ be a smooth bounded pseudoconvex domain of finite type $2m$ with $m > 1$. Then* $\lim_{\lambda\to\infty} \mathcal{N}(\lambda)/\lambda^{m+1} = 0$.

The Weyl-type asymptotic formula for $\mathcal{N}(\lambda)$ for a strictly pseudoconvex domain in $\mathbb{C}^n$ was established by Metivier [47] via an analysis of the spectral kernel of the $\bar\partial$-Neumann Laplacian. The heat kernel of the $\bar\partial$-Neumann Laplacian on a strictly pseudoconvex domain, as well as that of the Kohn Laplacian on the boundary, were systematically studied in a series of papers by Stanton; Beals, Greiner and Stanton; Stanton and Tartakoff; and Beals and Stanton (see [57,1, 58,2,3] and references therein). Metivier's formula was recovered as a consequence. Recently, the heat kernel of the Kohn Laplacian on a finite type boundary in $\mathbb{C}^2$ was studied by Nagel and Stein [49], from which it is plausible that one could also deduce a result similar to Theorem 1.2 for the Kohn Laplacian on the boundary.

It is noteworthy that the above result starkly contrasts with strictly pseudoconvex case [47]. When Ω is strictly pseudoconvex (i.e., $m = 1$), $\lim_{\lambda\to\infty} \mathcal{N}(\lambda)/\lambda^{1+m} = K$, where the positive constant K accounts for contributions from the interior as well as from the boundary. When Ω is of finite type $2m$ with $m > 1$, the contribution from the boundary dominates that from the interior (see the remark at the end of Section 6).

We prove Theorem 1.2 by studying the spectral kernel of the $\bar\partial$-Neumann Laplacian, following [47]. Our analysis also builds upon Kohn's subelliptic estimate for the $\bar\partial$-Neumann Laplacian [38], as well as the work of Catlin [10], Christ [12], Fefferman and Kohn [23], McNeal [46], and Nagel et al. [48] on the Bergman and Szegö kernels for a finite type domain in $\mathbb{C}^2$. There is one important distinction between the spectral kernel and the Bergman kernel: While the Bergman kernel transforms well under biholomorphic mappings, the spectral kernel does not. It does not transform well even under anisotropic dilations. To overcome this difficulty, instead of (locally) rescaling the domain to unit scale and studying the $\bar\partial$-Neumann Laplacian on the rescaled domain as in the Bergman kernel case, we rescale both the domain and the $\bar\partial$-Neumann Laplacian as in [47] (see Section 4). In doing so, we are led to anisotropic bidiscs that have larger radii in the complex normal direction. Roughly speaking, at a boundary point of type $2m$, the ratio of the radii in the complex tangential and normal directions for the bidiscs used here is $\tau : \tau^m$ whereas in the Bergman kernel case it is $\tau : \tau^{2m}$ ($\tau > 0$ is small). To establish desirable properties, such as doubling and engulfing properties, for these anisotropic bidiscs, we employ both the pseudoconvexity and finite type assumptions on the domain Ω (see Section 3). Note that only the finite type condition was used to establish these properties for the smaller bidiscs in the Bergman kernel case. Here in our analysis of the bidiscs, we make essential use of a result by Fornæss and Sibony [25]. Also crucial in our analysis is a Kohn-type uniform subelliptic estimate for the rescaled $\bar\partial$-Neumann Laplacian (see Lemma 4.5 below). After carefully flattening the boundary, we then reduce the problem to estimating eigenvalues of auxiliary operators on the half-space (see Section 5). The underpinning analysis here is a semi-classical analysis of Schrödinger operators with finitely degenerated magnetic field (compare [29,15]). The final steps of the proof of Theorem 1.2 are given in Section 6.

For the necessity, we establish the following result.

Theorem 1.3. *Let $\Omega \subset \mathbb{C}^2$ be a smooth bounded pseudoconvex domain. If there exist positive constants M and C such that $\mathcal{N}(\lambda) \leqslant C\lambda^M$, then $b\Omega$ is of finite type $\leqslant 8M$.*

In fact, we will prove a slight generalization of the above result in Section 7. An ingredient in the proof of Theorem 1.3 is a wavelet construction of Lemarié and Meyer[2] [44] (see Section 7). A result similar to Theorem 1.3 for the Kohn Laplacian on the boundaries in $\mathbb{C}^2$ is known to M. Christ [14].

Throughout the paper, we use C to denote positive constants which may be different in different appearances. We also use $F \lesssim G$ to denote $F \leqslant CG$ for some constant C independent of parameters manifest from the context, and use $F \approx G$ to denote $F \lesssim G \lesssim F$. For the reader's convenience, we make an effort to have our presentation self-contained.

2. Preliminaries

We first recall several relevant general facts from operator theory. Let H_1, H_2 be complex Hilbert spaces. Let T be a compact operator from H_1 into H_2. Then it follows from the min–max principle that the singular values of T (i.e., the non-zero eigenvalues of $|T| = (T^*T)^{1/2}$) are given by

$$s_j(T) = \inf_{g_1,\dots,g_{j-1}\in H_1} \sup\bigl\{\|Tf\| \bigm| f \in H_1,\ f \perp g_1,\dots,g_{j-1},\ \|f\| = 1\bigr\},\quad j = 1, 2, \dots, \tag{2.1}$$

where the singular values are arranged in decreasing order and repeated according to multiplicity (e.g., [61]). In particular, $s_1(T) = \|T\|$. (Throughout this paper, we will use $s_j(T)$ to denote the jth-singular value of a compact operator T arranged in this order.) It follows that for compact operators $T_1, T_2 : H_1 \to H_2$ and $T_3 : H_2 \to H_3$,

$$s_{j+k+1}(T_1 + T_2) \leqslant s_{j+1}(T_1) + s_{k+1}(T_2) \tag{2.2}$$

and

$$s_{j+k+1}(T_3 \circ T_1) \leqslant s_{j+1}(T_1)s_{k+1}(T_3). \tag{2.3}$$

Let Q be a non-negative, densely defined, and closed sesquilinear form on a complex Hilbert space H. Then Q uniquely determines a non-negative, densely defined, and self-adjoint (unbounded) operator S such that

$$Q(u, v) = (Su,\ v) = \bigl(S^{1/2}u, S^{1/2}v\bigr) \quad \text{and} \quad \mathrm{Dom}\bigl(S^{1/2}\bigr) = \mathrm{Dom}(Q)$$

for all $u \in \mathrm{Dom}(S)$ and $v \in \mathrm{Dom}(Q)$. (See, for example, [21].) For any subspace $L \subset \mathrm{Dom}(Q)$, let

$$\lambda(L) = \sup\bigl\{Q(u, u) \bigm| u \in L,\ \|u\| = 1\bigr\}. \tag{2.4}$$

[2] One could also use the compactly supported wavelets constructed by Daubechies (see [20]).

For any positive integer j, let

$$\lambda_j(Q) = \inf\{\lambda(L) \mid L \subset \mathrm{Dom}(Q),\ \dim(L) = j\}. \tag{2.5}$$

It follows that the associated operator S has compact resolvent if and only if $\lambda_j(Q) \to \infty$ as $j \to \infty$. In this case, $\lambda_j(Q)$ equals $\lambda_j(S)$, the jth eigenvalue of S, arranged in increasing order and repeated according to multiplicity. (Hereafter, for any non-negative, densely defined, self-adjoint operator S, we will use $\lambda_j(S)$ to denote the quantity given by (2.5) where Q is the sesquilinear form associated with S.)

We now recall the setup for the $\bar\partial$-Neumann Laplacian (e.g., [24,11]). Let Ω be a bounded domain in $\mathbb{C}^n$. For $1 \leqslant q \leqslant n$, let $L^2_{(0,q)}(\Omega)$ denote the space of $(0,q)$-forms with square integrable coefficients and with the standard Euclidean inner product whose norm is given by

$$\left\| \sum{}' a_J\, d\bar z_J \right\|^2 = \sum{}' \int_\Omega |a_J|^2\, dV(z),$$

where the prime indicates the summation over strictly increasing q-tuples J. (We consider a_J to be defined on all q-tuples, antisymmetric with respect to J.) For $0 \leqslant q \leqslant n-1$, let $\bar\partial_q : L^2_{(0,q)}(\Omega) \to L^2_{(0,q+1)}(\Omega)$ be the $\bar\partial$-operator defined in the sense of distribution. This is a closed and densely defined operator. Let $\bar\partial^*_q$ be its adjoint. For $1 \leqslant q \leqslant n-1$, let

$$Q_q(u,v) = (\bar\partial_q u, \bar\partial_q v) + \left(\bar\partial^*_{q-1} u, \bar\partial^*_{q-1} v\right)$$

be the sesquilinear form on $L^2_{(0,q)}(\Omega)$ with $\mathrm{Dom}(Q_q) = \mathrm{Dom}(\bar\partial_q) \cap \mathrm{Dom}(\bar\partial^*_{q-1})$. It is evident that Q_q is non-negative, densely defined, and closed. The operator associated with Q_q is the $\bar\partial$-Neumann Laplacian $\Box_q$ on $L^2_{(0,q)}(\Omega)$.

The following lemma is a simple consequence of the min–max principle (2.5).

Lemma 2.1. *Suppose $\lambda_k(Q) \geqslant C_1 k^\varepsilon$, $1 \leqslant k \leqslant j$, for some constants $C_1 > 0$ and $\varepsilon > 0$. If $u_k \in \mathrm{Dom}(Q)$, $1 \leqslant k \leqslant j$, satisfy*

$$\left\| \sum_{k=1}^{j} c_k u_k \right\|^2 \geqslant C_2 \sum_{k=1}^{j} |c_k|^2$$

for some constant $C_2 > 0$ and for all $(c_1, \ldots, c_j) \in \mathbb{C}^j$, then

$$\max_{1\leqslant k\leqslant j} Q(u_k, u_k) \geqslant C_1 C_2 j^\varepsilon / (1+\varepsilon).$$

Proof. Let $\tilde\lambda_k$, $1 \leqslant k \leqslant j$, be the eigenvalues of the Hermitian matrix $M = (Q(u_k, u_l))_{1\leqslant k,l\leqslant j}$. Then by the min–max principle,

$$\tilde\lambda_k = \inf\{\tilde\lambda(\tilde L) \mid \tilde L \subset \mathbb{C}^j,\ \dim(\tilde L) = k\}$$

where

$$\tilde{\lambda}(\tilde{L}) = \sup\left\{ \sum_{k,l=1}^{j} c_k \bar{c}_l Q(u_k, u_l) \;\middle|\; (c_1, \ldots, c_j) \in \tilde{L},\ \sum_{k=1}^{j} |c_k|^2 = 1 \right\}.$$

Let $L = \{\sum_{l=1}^{j} c_l u_l \mid (c_1, \ldots, c_j) \in \tilde{L}\}$. Then $\tilde{\lambda}(\tilde{L}) \geqslant C_2 \lambda(L)$, where $\lambda(L)$ is given by (2.4). Hence $\tilde{\lambda}_k \geqslant C_2 \lambda_k(Q)$ for all $1 \leqslant k \leqslant j$. Therefore,

$$\begin{aligned} j \max_{1\leqslant k\leqslant j} Q(u_k, u_k) \geqslant \operatorname{tr}(M) = \sum_{k=1}^{j} Q(u_k, u_k) = \sum_{k=1}^{j} \tilde{\lambda}_k \geqslant C_2 \sum_{k=1}^{j} \lambda_k(Q) \\ \geqslant C_1 C_2 \sum_{k=1}^{j} k^{\varepsilon} \geqslant C_1 C_2 \int_0^j x^{\varepsilon}\, dx = C_1 C_2 j^{\varepsilon+1}/(\varepsilon + 1). \end{aligned}$$

Dividing both sides by j, we obtain the lemma. □

Proposition 2.2. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$. Then $\lambda_j(\Box_q) \leqslant \lambda_k(\Box_{q+1})$ for all positive integers j, q, and k such that $1 \leqslant q \leqslant n-1$ and $n(j-1) < k \leqslant nj$. In particular, if $\Box_q$ has compact resolvent, so is $\Box_{q+1}$.*

Proof. Let $u = \sum'_{|J|=q+1} u_J\, d\bar{z}_J \in C^{\infty}(\overline{\Omega}) \cap \operatorname{Dom}(Q_{q+1})$. Write

$$u = \frac{1}{(q+1)!} \sum_{|J|=q+1} u_J\, d\bar{z}_J = \frac{1}{q+1} \sum_{j=1}^{n} \left(\frac{(-1)^q}{q!} \sum_{|K|=q} u_{jK}\, d\bar{z}_K \right) \wedge d\bar{z}_j = \frac{1}{q+1} \sum_{j=1}^{n} u_j \wedge d\bar{z}_j$$

where the u_j's are $(0, q)$-forms defined by the expression in the parenthesis in the above equalities. It is easy to see that $u_j \in C^{\infty}(\overline{\Omega}) \cap \operatorname{Dom}(Q_q)$ and $\sum_{j=1}^{n} \|u_j\|^2 = (q+1)\|u\|^2$. Moreover, by the Kohn–Morrey formula, we have

$$\begin{aligned} \sum_{j=1}^{n} Q_q(u_j, u_j) &= (q+1) \sum_{|J|=q+1}' \sum_{j=1}^{n} \int_{\Omega} \left| \frac{\partial u_J}{\partial \bar{z}_j} \right|^2 dV + q \sum_{|K|=q}' \sum_{j,k=1}^{n} \int_{b\Omega} \frac{\partial^2 \rho(z)}{\partial z_j \partial \bar{z}_k} u_{jK} \bar{u}_{kK}\, dS \\ &\leqslant (q+1) Q_{q+1}(u, u) \end{aligned} \tag{2.6}$$

where ρ is any defining function of Ω whose gradient has unit length on $b\Omega$. Consider $\widetilde{Q}(u, u) = \sum_{j=1}^{n} Q_q(u_j, u_j)$ as a quadratic form on $\bigoplus_{j=1}^{n} L^2_{(0,q)}(\Omega)$ with domain $\bigoplus_{j=1}^{n} \operatorname{Dom}(Q_q)$. The associated self-adjoint operator is then $\widetilde{\Box} = \bigoplus_{j=1}^{n} \Box_q$. Let $\tilde{\lambda}_j$ be the quantity defined by (2.5) with Q replaced by $\widetilde{Q}$. From (2.5), we then have $\lambda_j(\Box_{q+1}) \geqslant \tilde{\lambda}_j$. If $\Box_q$ has compact resolvent, so does $\widetilde{\Box}$. In this case, $\tilde{\lambda}_k = \lambda_j(\Box_q)$ and hence $\lambda_k(\Box_{q+1}) \geqslant \lambda_j(\Box_q)$ for $n(j-1) < k \leqslant nj$. If $\Box_q$ does not have compact resolvent, let a be the bottom of its essential spectrum. If $\lambda_j(\Box_q) < a$ for all positive integers j, then the $\lambda_j(\Box_q)$'s are again eigenvalues of finite multiplicity and $\tilde{\lambda}_k = \lambda_j(\Box_q)$ for $n(j-1) < k \leqslant nj$. Otherwise, let j_0 be the smallest integer such that $\lambda_{j_0}(\Box_q) = a$. In this case, $\lambda_j(\Box_q)$, $1 \leqslant j < j_0$, are eigenvalues and $\lambda_j(\Box_q) = a$ for all $j \geqslant j_0$. Hence $\tilde{\lambda}_k = \lambda_j(\Box_q)$ for $1 \leqslant j < j_0$ and $n(j-1) < k \leqslant nj$, and $\tilde{\lambda}_k = a$ for $k > nj_0$. Therefore, in all cases, we have $\lambda_j(\Box_q) = \tilde{\lambda}_k$ for $n(j-1) < k \leqslant nj$. We thus conclude the proof of the first part of the lemma. The last statement of the lemma is then a direct consequence of the fact from operator theory that $\Box_q$ has compact resolvent if and only if $\lambda_j(\Box_q) \to \infty$ as $j \to \infty$ (see for instance [21]). $\Box$

Next we recall elements of a wavelet construction of Lemarié and Meyer ([44]; see also [20,32]). Let $a(t) \in C_0^\infty(-\infty, \infty)$ be the cut-off function defined by $a(t) = \exp(1/(t-1) - 2/(2t-1))$ on $(1/2, 1)$ and $a(t) = 0$ elsewhere. Let

$$b(t) = \begin{cases} \left(\int_{1/2}^{1+t} a(s)\,ds / \int_{1/2}^{1} a(s)\,ds\right)^{1/2}, & \text{if } t < 0, \\ \left(\int_{t}^{1} a(s)\,ds / \int_{1/2}^{1} a(s)\,ds\right)^{1/2}, & \text{if } t \geqslant 0. \end{cases}$$

Then $b(t)$ is a smooth function supported in $[-1/2, 1]$ and satisfying $b(t) \equiv 1$ on $[0, 1/2]$ and $b^2(t) + b^2(t-1) \equiv 1$ on $[1/2, 1]$. It has been shown by Lemarié and Meyer that for any function b with these properties, its Fourier transform $\hat{b}(\zeta)$ is a wavelet on $\mathbb{R}$. In particular, we have the following:

Lemma 2.3. *(See [44].) Let c_0 be the L^2-norm of $b(t)$ and let*

$$g_{k,c}(t) = c_0^{-1} c^{1/2} b(ct) e^{2\pi kcti}.$$

Then for any given $c > 0$, $\{g_{k,c}(t)\}_{k\in\mathbb{Z}}$ is an orthonormal sequence in $L^2(\mathbb{R})$.

Proof. We provide the proof for completeness. It is easy to see that the L^2-norm of $g_{k,c}(t)$ is 1. For any distinct $k, k' \in \mathbb{Z}$, we have

$$\begin{aligned}
(g_{k,c},\ g_{k',c}) &= c_0^{-2} \int_{-\infty}^{\infty} b^2(t) e^{2\pi(k-k')ti}\,dt \\
&= c_0^{-2}\left(\int_{-1/2}^{0} + \int_{0}^{1/2} + \int_{1/2}^{1}\right) b^2(t) e^{2\pi(k-k')ti}\,dt \\
&= c_0^{-2}\left(\int_{1/2}^{1} \big(b^2(t-1) + b^2(t)\big) e^{2\pi(k-k')ti}\,dt + \int_{0}^{1/2} b^2(t) e^{2\pi(k-k')ti}\,dt\right) \\
&= c_0^{-2} \int_{0}^{1} e^{2\pi(k-k')ti}\,dt = 0. \qquad \Box
\end{aligned}$$

3. Special holomorphic coordinates and anisotropic bidiscs

The anisotropic geometry of finite type boundaries in $\mathbb{C}^2$ has been studied in depth (see, for examples, [50,48,10]). The key difference here is, as noted above, that by taking into account the pseudoconvexity assumption of the boundary and using a result of Fornæss and Sibony [25], we establish desirable properties for anisotropic bidiscs with larger size in the complex normal direction.

Let $\Omega = \{z \in \mathbb{C}^2 \mid r(z) < 0\}$ be a smooth bounded domain with a defining function $r \in C^\infty(\mathbb{C}^2)$. Assume that $|dr| = 1$ on $b\Omega$. Let

$$L = \frac{\partial r}{\partial z_2}\frac{\partial}{\partial z_1} - \frac{\partial r}{\partial z_1}\frac{\partial}{\partial z_2}.$$

Let $z' \in b\Omega$. For $j, k \geqslant 1$, let

$$\mathcal{L}_{jk}\partial\bar\partial r(z') = \underbrace{L \ldots L}_{j-1 \text{ times}}\ \underbrace{\bar L \ldots \bar L}_{k-1 \text{ times}}\ \partial\bar\partial r(L, \bar L)(z').$$

Let m be any positive integer. For any $2 \leqslant l \leqslant 2m$, let

$$A_l(z') = \Bigl(\sum_{\substack{j+k\leqslant l\\ j,k>0}} \bigl|\mathcal{L}_{jk}\partial\bar\partial r(z')\bigr|^2\Bigr)^{1/2}. \tag{3.1}$$

Let $\tilde r$ be any defining function for Ω and let $\widetilde{L}$ be any non-vanishing complex tangential vector field of $b\Omega$. Let $\tilde A_l(z')$ be defined by (3.1) with r replaced by $\tilde r$ and L by $\widetilde L$, then it is easy to check that $A_l(z') \approx \tilde A_l(z')$. Thus, whether or not $A_l(z')$ vanishes is a property that is independent of the choice of either the defining function or the complex tangential vector field of $b\Omega$. Furthermore, $b\Omega$ is of finite type $2m'$ at z' if and only if $A_l(z')$ is 0 for all $2 \leqslant l \leqslant 2m' - 1$ but is positive for $l = 2m'$. For any $\tau > 0$, let

$$\delta(z', \tau) = \sum_{l=2}^{2m} A_l(z')\tau^l. \tag{3.2}$$

Evidently,

$$\delta(z', \tau) \lesssim \tau^2 \quad \text{and} \quad c^{2m}\delta(z', \tau) \leqslant \delta(z', c\tau) \leqslant c^2\delta(z', \tau), \tag{3.3}$$

for any τ and c such that $0 < \tau, c < 1$. Furthermore, $b\Omega$ is of finite type $2m$ if and only if $\delta(z', \tau) \gtrsim \tau^{2m}$ uniformly for all $z' \in b\Omega$ and $\delta(z'_0, \tau) \lesssim \tau^{2m}$ for some $z'_0 \in b\Omega$.

Let U be a neighborhood of a boundary point. Assume without loss of generality that $|\partial r/\partial z_2| \gtrsim 1$ on U. After a change of (global) holomorphic coordinates of the form

$$(\xi_1, \xi_2) = \Phi'(z_1, z_2) = \Bigl(z_1 - z'_1,\ 2\frac{\partial r}{\partial z_2}(z')\bigl(z_2 - z'_2\bigr) + \sum_{k=1}^{2m} \alpha_k(z')\bigl(z_1 - z'_1\bigr)^k\Bigr), \tag{3.4}$$

we have that the image $\Omega' = \Phi'(\Omega)$ is defined by

$$\rho(\xi) = r\bigl((\Phi')^{-1}(\xi)\bigr) = \operatorname{Re}\xi_2 + \sum_{\substack{2\leqslant j+k\leqslant 2m\\ j,k>0}} a_{jk}(z')\xi_1^j\bar\xi_1^k + O\bigl(|\xi_1|^{2m+1} + |\xi_2||\xi|\bigr). \tag{3.5}$$

The $\alpha_k(z')$'s and $a_{jk}(z')$'s depend smoothly on z', and they are unique in the sense that if after a holomorphic change of coordinates $\check{\Phi}'$ of the form (3.4) but with possible different $\alpha_k(z')$'s, $r((\check{\Phi}')^{-1}(\xi))$ is in the form of (3.5) but with possible different $a_{jk}(z')$'s, then $\Phi' = \check{\Phi}'$. (See [10] for a detailed discussion on the above coordinates.) Solving $\operatorname{Re}\xi_2$ in terms of the other variables, we have that $b\Omega'$ is defined near the origin by $\tilde{\rho}(\xi) = \operatorname{Re}\xi_2 + \tilde{h}(\xi_1, \operatorname{Im}\xi_2)$, where

$$\tilde{h}(\xi_1, \operatorname{Im}\xi_2) = \sum_{\substack{2\leqslant j+k\leqslant 2m\\ j,k>0}} \tilde{a}_{jk}(z')\xi_1^j\bar{\xi}_1^k + O\big(|\xi_1|^{2m+1} + |\operatorname{Im}\xi_2||\xi_1| + |\operatorname{Im}\xi_2|^2\big). \tag{3.6}$$

It is easy to see that

$$A_l(z') \approx \sum_{\substack{j+k\leqslant l\\ j,k>0}} \big|a_{jk}(z')\big| \approx \sum_{\substack{j+k\leqslant l\\ j,k>0}} \big|\tilde{a}_{jk}(z')\big|, \tag{3.7}$$

for $2 \leqslant l \leqslant 2m$.

Write

$$\tilde{h}(\xi_1, \operatorname{Im}\xi_2) = \widetilde{P}(\xi_1) + (\operatorname{Im}\xi_2)\widetilde{Q}(\xi_1) + O\big(|\xi_1|^{2m+1} + |\operatorname{Im}\xi_2|^2 + |\operatorname{Im}\xi_2||\xi_1|^{m+1}\big), \tag{3.8}$$

where

$$\widetilde{P}(\xi_1) = \sum_{\substack{2\leqslant j+k\leqslant 2m\\ j,k>0}} \tilde{a}_{jk}(z')\xi_1^j\bar{\xi}_1^k, \qquad \widetilde{Q}(\xi_1) = \sum_{1\leqslant j+k\leqslant m} \tilde{b}_{jk}(z')\xi_1^j\bar{\xi}_1^k.$$

The harmonic terms can be expunged from the polynomial $\widetilde{Q}$ without introducing harmonic terms into the $\widetilde{P}$ term by a change of (local) holomorphic coordinates of the form

$$(\tilde{\xi}_1, \tilde{\xi}_2) = \Phi^*(\xi_1, \xi_2) = \left(\xi_1, \xi_2\prod_{j=1}^{m}\big(1 - \beta_j(z')\xi_1^j\big)\right).$$

(See [25].) Finally, after another change of coordinates of the form (compare [22]):

$$(\zeta_1, \zeta_2) = \widehat{\Phi}(\tilde{\xi}_1, \tilde{\xi}_2) = \big(\tilde{\xi}_1, \tilde{\xi}_2 - \gamma(z')(\tilde{\xi}_2)^2\big),$$

we can also eliminate the term $\gamma(z')(\operatorname{Im}\tilde{\xi}_2)^2$ from the remainder of the Taylor expansion without introducing harmonic terms into the $\widetilde{P}$ and $\widetilde{Q}$ terms, and obtain that $b\Omega$ is defined near z' in the new (ζ_1, ζ_2)-coordinates by $\hat{\rho} = \operatorname{Re}\zeta_2 + \hat{h}(\zeta_1, \operatorname{Im}\zeta_2) = 0$ with

$$\hat{h}(\zeta_1, \operatorname{Im}\zeta_2) = \widehat{P}(\zeta_1) + (\operatorname{Im}\zeta_2)\widehat{Q}(\zeta_1) + O\big(|\zeta_1|^{2m+1} + |\operatorname{Im}\zeta_2||\zeta_1|^{m+1} + |\operatorname{Im}\zeta_2|^2|\zeta_1|\big), \tag{3.9}$$

where

$$\widehat{P}(\zeta_1)=\sum_{l=2}^{2m}\widehat{P}_l(\zeta_1)=\sum_{l=2}^{2m}\sum_{\substack{j+k=l\\ j,k>0}}\hat{a}_{jk}(z')\zeta_1^j\bar{\zeta}_1^k$$

and

$$\widehat{Q}(\zeta_1)=\sum_{l=2}^{m}\widehat{Q}_l(\zeta_1)=\sum_{l=2}^{m}\sum_{\substack{j+k=l\\ j,k>0}}\hat{b}_{jk}(z')\zeta_1^j\zeta_1^k.$$

Let

$$\widehat{A}_l(z')=\Bigg(\sum_{\substack{j+k\leqslant l\\ j,k>0}}\big|\hat{a}_{jk}(z')\big|^2\Bigg)^{1/2}\quad\text{and}\quad\widehat{B}_l(z')=\Bigg(\sum_{\substack{j+k\leqslant l\\ j,k>0}}\big|\hat{b}_{jk}(z')\big|^2\Bigg)^{1/2}.$$

We now summarize what we have obtained from these changes of holomorphic coordinates. For any $z'\in U\cap b\Omega$, there exists a neighborhood $U_{z'}$ of z' and a biholomorphic map $\zeta=\widehat{\Psi}_{z'}(z)=\widehat{\Phi}\circ\Phi^*\circ\Phi'(z)$ from $U_{z'}$ onto a ball $B(0,\varepsilon_0)$ of uniform radius ε_0 such that

(A-1) $\widehat{\Psi}_{z'}$ depends smoothly on z' and its components are holomorphic polynomials of degrees $\leqslant m^2+5m$ for each z'. Moreover, the Jacobian determinant $J\widehat{\Psi}_{z'}$ of $\widehat{\Psi}_{z'}$ is uniformly bounded from above and below on $U_{z'}$.

(A-2) $\widehat{\Psi}_{z'}(z')=0$ and $\widehat{\Psi}_{z'}(U_{z'}\cap\Omega)=\{\zeta\in B(0,\varepsilon_0)\mid\hat{\rho}(\zeta)=\operatorname{Re}\zeta_2+\hat{h}(\zeta_1,\operatorname{Im}\zeta_2)<0\}$, where $\hat{h}(\zeta_1,\operatorname{Im}\zeta_2)$ is in the form of (3.9).

(A-3) There exist positive constants C_1 and C_2 independent of z' such that $C_1A_l(z')\leqslant\widehat{A}_l(z')\leqslant C_2A_l(z')$ for $2\leqslant l\leqslant 2m$.

Notice that these properties hold for any smooth bounded domain Ω. From now on, we will assume that Ω is pseudoconvex of finite type $2m$. When these assumptions come into play, then it follows from [25] that

$$\sum_{l=2}^{m}\|\widehat{Q}_l\|_\infty|\zeta_1|^l\lesssim|\zeta_1|\Bigg(\sum_{l=2}^{2m}\|\widehat{P}_l\|_\infty|\zeta_1|^l\Bigg)^{1/2},\tag{3.10}$$

where $\|P\|_\infty$ denotes the sup-norm of $P(\zeta_1)$ on $|\zeta_1|=1$. It is easy to see that

$$\Bigg\|\sum_{j+k=l}c_{jk}\zeta_1^j\bar{\zeta}_1^k\Bigg\|_\infty\approx\sum_{j+k=l}|c_{jk}|.$$

Therefore, in light of (A-3) and (3.10), we have

$$\sum_{l=2}^{m}\widehat{B}_l(z')\tau^l\lesssim\tau\big(\delta(z',\tau)\big)^{1/2}\tag{3.11}$$

for $0<\tau<1$.

Let

$$\widehat{P}_\tau(z') = \{|\zeta_1| < \tau,\ |\zeta_2| < \delta(z',\tau)^{1/2}\},$$

and let

$$R_\tau(z') = (\widehat{\Psi}_{z'})^{-1}\big(\widehat{P}_\tau(z')\big).$$

We now study the anisotropic "bidiscs" $R_\tau(z')$. Notice that the size of $\widehat{P}_\tau(z')$ is different from those used to study the Bergman kernel in [10,46,48]. Here we have $|\zeta_2| < (\delta(z',\tau))^{1/2}$ instead of $|\zeta_2| < \delta(z',\tau)$. This seems to be crucial in our analysis. Let τ_0 be a sufficiently small positive constant such that $\widehat{P}_{\tau_0}(z') \subset B(0,\varepsilon_0)$ for all $z' \in b\Omega$. Let

$$\begin{aligned}\widehat{\mathcal{S}}^{a,b}_{z'} = \big\{ f \in C^\infty\big(B(0,\varepsilon_0)\big) \ \big|\ &\forall j,k \geqslant 0,\ \exists C_{jk} > 0 \text{ such that} \\ &\big|D^j_{\zeta_1} D^k_{\zeta_2} f(\zeta)\big| \leqslant C_{jk}\tau^{a-j}\big(\delta(z',\tau)\big)^{b-k/2},\ \forall \zeta \in \widehat{P}_\tau(z'),\ \forall \tau \in (0,\tau_0)\big\}. \end{aligned} \tag{3.12}$$

(Here $D^j_{\zeta_l}$ denotes the partial derivatives of order j with respect to ζ_l or $\bar\zeta_l$.) The following facts can be checked easily:

(B-1) If $f \in \widehat{\mathcal{S}}^{a,b}_{z'}$, then $D^j_{\zeta_1} D^k_{\zeta_2} f \in \widehat{\mathcal{S}}^{a-j,b-k/2}_{z'}$.
(B-2) If $f \in \widehat{\mathcal{S}}^{a,b}_{z'}$ and $g \in \widehat{\mathcal{S}}^{c,d}_{z'}$, then $fg \in \widehat{\mathcal{S}}^{a+c,b+d}_{z'}$.

Lemma 3.1. *Under the pseudoconvexity and finite type assumptions on Ω, $\hat h(\zeta_1, \operatorname{Im}\zeta_2) \in \widehat{\mathcal{S}}^{0,1}_{z'}$.*

Proof. Write $\delta = \delta(z',\tau)$ and let $\hat h_\tau(w_1,w_2) = (1/\delta)\hat h(\tau w_1, \delta^{1/2}\operatorname{Im} w_2)$. Since $\tau^{2m} \lesssim \delta \lesssim \tau^2$, it follows that the Taylor expansion of $\hat h_\tau$ at the origin has the form

$$\begin{aligned}\hat h_\tau(w_1,w_2) = \sum_{l=2}^{2m}\sum_{j+k=l}\frac{\tau^l \hat a_{jk}(z')}{\delta} w_1^j \bar w_1^k + (\operatorname{Im} w_2)\sum_{l=2}^{m}\sum_{j+k=l}\frac{\tau^l \hat b_{jk}(z')}{\delta^{1/2}} w_1^j \bar w_1^k \\ + O\big(\tau\big(|w_1|^{2m+1} + |\operatorname{Im} w_2||w_1|^{m+1} + |\operatorname{Im} w_2|^2|w_1|\big)\big).\end{aligned} \tag{3.13}$$

Notice that the Taylor coefficients in the first sum above have modulus $\lesssim 1$ by property (A-3) and those in the second sum have modulus $\lesssim \tau$ by (3.11). The coefficients of the Taylor expansion of the remainder are also $\lesssim \tau$ as shown above. Therefore, $\hat h(\zeta_1, \operatorname{Im}\zeta_2) \in \widehat{\mathcal{S}}^{0,1}_{z'}$. $\square$

The next two lemmas establish the doubling and engulfing properties for the anisotropic (biholomorphic images of) bidiscs $R_\tau(z')$. Their proofs are similar to those of Propositions 1.3 and 1.7 in [10].

Lemma 3.2. *Under the same assumptions and for sufficiently small $\tau > 0$, if $z'' \in R_\tau(z') \cap b\Omega$, then $\delta(z'',\tau) \approx \delta(z',\tau)$.*

Proof. We shall use the above lemma. Let $\hat{r}(z) = \hat{\rho}(\widehat{\Psi}_{z'}^{-1}(z))$. Let

$$L^* = \frac{\partial}{\partial \zeta_1} - \frac{\partial \hat{\rho}}{\partial \zeta_1}\left(\frac{\partial \hat{\rho}}{\partial \zeta_2}\right)^{-1}\frac{\partial}{\partial \zeta_2} = \frac{\partial}{\partial \zeta_1} - \frac{\partial \hat{h}}{\partial \zeta_1}\left(\frac{1}{2} + \frac{\partial \hat{h}}{\partial \zeta_2}\right)^{-1}\frac{\partial}{\partial \zeta_2}.$$

Notice that the coefficient of $\partial/\partial\zeta_2$ in the above expression of L^* belongs to $\mathcal{S}_{z'}^{-1,1}$. Let $L' = (\widehat{\Psi}_{z'}^{-1})_*(L^*)$. Write

$$\mathcal{L}'_{j,k}\partial\bar{\partial}\hat{r}(z) = \underbrace{L' \dots L'}_{j-1 \text{ times}}\,\underbrace{\overline{L'} \dots \overline{L'}}_{k-1 \text{ times}}\partial\bar{\partial}\hat{r}(L', \overline{L'})$$

and

$$\mathcal{L}^*_{j,k}\partial\bar{\partial}\hat{\rho}(\zeta) = \underbrace{L^* \cdots L^*}_{j-1 \text{ times}}\,\underbrace{\overline{L^*} \dots \overline{L^*}}_{k-1 \text{ times}}\partial\bar{\partial}\hat{\rho}\big(L^*, \overline{L^*}\big).$$

Then for $z \in U_{z'}$,

$$\mathcal{L}'_{j,k}\partial\bar{\partial}\hat{r}(z) = \mathcal{L}^*_{j,k}\partial\bar{\partial}\hat{\rho}(\zeta),$$

where $\zeta = \widehat{\Psi}_{z'}(z)$. It is easy to see that $\mathcal{L}^*_{j,k}\partial\bar{\partial}\hat{\rho}(\zeta) \in \mathcal{S}_{z'}^{-j-k,1}$; in fact,

$$\mathcal{L}^*_{j,k}\partial\bar{\partial}\hat{\rho}(\zeta) = \frac{\partial^{j+k}\hat{h}(\zeta)}{\partial\zeta_1^j\partial\bar{\zeta}_1^k} + s_{-j-k+1}(\zeta) \tag{3.14}$$

for some $s_{-j-k+1}(\zeta) \in \mathcal{S}_{z'}^{-j-k+1,1}$. Since $z'' \in R_\tau(z') \cap b\Omega$, $\zeta'' = \widehat{\Psi}_{z'}(z'') \in \widehat{P}_\tau(z')$. Hence

$$\begin{aligned} A_l(z'') &\approx \max\big\{\big|\mathcal{L}'_{j,k}\partial\bar{\partial}\hat{r}(z'')\big|;\ 2 \leqslant j+k \leqslant l\big\} \\ &= \max\big\{\big|\mathcal{L}^*_{j,k}\partial\bar{\partial}\hat{\rho}(\zeta'')\big|;\ 2 \leqslant j+k \leqslant l\big\} \\ &\lesssim \tau^{-l}\delta(z', \tau). \end{aligned}$$

Therefore, $\delta(z'', \tau) \lesssim \delta(z', \tau)$. We now prove the estimate in the opposite direction. From the definition of $\delta(z', \tau)$ we know that there exist $j_0, k_0 > 0$ with $j_0 + k_0 = l_0 \leqslant 2m$ such that

$$\big|\mathcal{L}^*_{j_0,k_0}\partial\bar{\partial}\hat{\rho}(0)\big| \gtrsim \tau^{-l_0}\delta(z', \tau),$$

where the constant in the above estimate is independent of z' and z''. Now let $z'' \in R_{\varepsilon\tau}(z')$ where ε is a sufficiently small constant to be determined. By (3.14) and (3.3), we have

$$\begin{aligned} \big|\mathcal{L}^*_{j_0,k_0}\partial\bar{\partial}\hat{\rho}(\zeta'') - \mathcal{L}^*_{j_0,k_0}\partial\bar{\partial}\hat{\rho}(0)\big| &\lesssim \left|\frac{\partial^{l_0}\hat{h}(\zeta'')}{\partial\zeta_1^{j_0}\partial\bar{\zeta}_1^{k_0}} - \frac{\partial^{l_0}\hat{h}(0)}{\partial\zeta_1^{j_0}\partial\bar{\zeta}_1^{k_0}}\right| + \tau^{-l_0+1}\delta(z', \tau) \\ &\lesssim \big(\tau^{-l_0-1}\delta(z', \tau)\big)\big|\zeta_1''\big| + \big(\tau^{-l_0}\delta(z', \tau)^{1/2}\big)\big|\zeta_2''\big| + \tau^{-l_0+1}\delta(z', \tau) \\ &\lesssim (\varepsilon + \tau)\tau^{-l_0}\delta(z', \tau). \end{aligned}$$

Therefore when both τ and ε are sufficiently small, we have

$$\delta(z'', \tau) \gtrsim \tau^{l_0} \left| \mathcal{L}_{j_0,k_0} \partial\bar{\partial}\hat{\rho}(\zeta'') \right| \gtrsim \delta(z', \tau).$$

We then conclude the proof by replacing $\varepsilon\tau$ by τ and using (3.3). $\square$

Lemma 3.3. *If $z'' \in R_\tau(z') \cap b\Omega$, then there exists a positive constant C such that*

$$R_\tau(z') \subset R_{C\tau}(z'') \quad \textit{and} \quad R_\tau(z'') \subset R_{C\tau}(z'). \tag{3.15}$$

Proof. It follows from $\Phi'(R_\tau(z')) = (\widehat{\Phi} \circ \Phi^*)^{-1}(\widehat{P}_\tau(z'))$ that

$$\left\{|\xi_1| < C^{-1}\tau,\ |\xi_2| < \delta\left(z', C^{-1}\tau\right)^{1/2}\right\} \subset \Phi'\left(R_\tau(z')\right) \subset \left\{|\xi_1| < C\tau,\ |\xi_2| < \delta(z', C\tau)^{1/2}\right\}$$

for some constant $C > 0$. Thus $\xi'' = \Phi'(z'') \in \{|\xi_1| < C\tau,\ |\xi_2| < \delta(z', C\tau)^{1/2}\}$. After a change of coordinates of the form

$$(\tilde{\xi}_1, \tilde{\xi}_2) = \Psi''(\xi_1, \xi_2) = \left(\xi_1 - \xi_1'',\ 2\frac{\partial\rho}{\partial\xi_2}(\xi'')\left(\xi_2 - \xi_2''\right) + \sum_{k=1}^{2m} e_k(z'')\left(\xi_1 - \xi_1''\right)^k\right), \tag{3.16}$$

we have

$$r\left((\Psi'' \circ \Phi')^{-1}(\tilde{\xi})\right) = \sum_{\substack{2\leqslant j+k\leqslant 2m \\ j,k>0}} a_{jk}(z'')\tilde{\xi}_1^j\bar{\tilde{\xi}}_1^k + O\left(|\tilde{\xi}_1|^{2m+1} + |\tilde{\xi}_1||\tilde{\xi}|\right).$$

The $e_k(z'')$'s are determined inductively as follows:

$$e_1(z'') = 2\frac{\partial\rho}{\partial\xi_1}(\xi''), \qquad e_k(z'') = \frac{2}{k!}\frac{\partial^k\rho_k}{\partial\xi_1^k}(0), \quad k \geqslant 2,$$

where

$$\rho_1 = \rho, \qquad \rho_k = \rho_{k-1} \circ (\phi_{k-1})^{-1}, \quad k \geqslant 2,$$

and

$$\phi_1 = \left(\xi_1 - \xi_1'', 2\frac{\partial\rho}{\partial\xi_2}(\xi'')\left(\xi_2 - \xi_2''\right) + e_1(z'')\left(\xi_1 - \xi_1''\right)\right),$$
$$\phi_k = \left(\xi_1, \xi_2 + e_k(z'')\xi_1^k\right), \quad k \geqslant 2.$$

It follows from (3.5) and (3.7) that

$$\left|D^l_{\xi_1}\rho_1(\xi)\right| \lesssim \tau^{-l}\left(\delta(z', \tau)\right)^{1/2}, \quad \text{for } |\xi_1| \lesssim \tau,\ |\xi_2| \lesssim \left(\delta(z', \tau)\right)^{1/2}.$$

By induction on k, we obtain that

$$\left|D^l_{\xi_1}\rho_k(\xi)\right| \lesssim \tau^{-l}\big(\delta(z',\tau)\big)^{1/2} \quad \text{and} \quad \left|e_k(z'')\right| \lesssim \tau^{-k}\big(\delta(z',\tau)\big)^{1/2}. \tag{3.17}$$

By the uniqueness in the sense noted after (3.5), we obtain as above that

$$\begin{aligned}\big\{|\tilde{\xi}_1| < C^{-1}\tau,\ |\tilde{\xi}_2| < \delta\big(z'', C^{-1}\tau\big)^{1/2}\big\} &\subset (\Psi'' \circ \Phi')\big(R_\tau(z'')\big)\\ &\subset \big\{|\tilde{\xi}_1| < C\tau,\ |\tilde{\xi}_2| < \delta(z'', C\tau)^{1/2}\big\}.\end{aligned}$$

It follows from Lemma 3.2, (3.17), and (3.16) that if $\xi \in \Phi'(R_\tau(z'))$, then

$$|\tilde{\xi}_1| \lesssim \tau \quad \text{and} \quad |\tilde{\xi}_2| \lesssim \big(\delta(z'',\tau)\big)^{1/2}.$$

Thus, $R_\tau(z') \subset R_{C\tau}(z'')$. Similarly, $R_\tau(z'') \subset R_{C\tau}(z')$. □

Denote by $d(z)$ the Euclidean distance from z to $b\Omega$. Let $\pi(z)$ be the projection from a neighborhood of $b\Omega$ onto $b\Omega$ such that $|z-\pi(z)| = d(z) \approx r(z)$. Denote by χ_A the characteristic function for a set A. Let

$$A_\tau = \big\{z \in \Omega \;\big|\; d(z) < \big(\delta\big(\pi(z),\tau\big)\big)^{1/2}\big\}.$$

The following lemma is an easy consequence of Lemmas 3.2 and 3.3.

Lemma 3.4. *For any $\alpha \in \mathbb{R}$, there exists a sufficiently large constant $C > 0$ such that for any sufficiently small $\tau > 0$ and for any $z \in A_\tau$,*

$$\chi_{A_{C^{-1}\tau}}(z) \lesssim \tau^{-2}\big(\delta\big(\pi(z),\tau\big)\big)^{-\alpha-\frac{1}{2}} \int\limits_{b\Omega} \chi_{R_\tau(z')\cap\Omega}(z)\big(\delta(z',\tau)\big)^{\alpha}\, dS(z') \lesssim \chi_{A_{C\tau}}(z).$$

Proof. It is easy to see that if $z \in R_\tau(z')$ then $\pi(z) \in R_{C\tau}(z')$. Thus by Lemma 3.2, $\delta(z',\tau) \approx \delta(\pi(z),\tau)$, and by Lemma 3.3, $z' \in R_{C\tau}(\pi(z))$. It follows that

$$A_{C^{-1}\tau} \subset \bigcup_{z' \in b\Omega} R_\tau(z') \cap \Omega \subset A_{C\tau},$$

and for any $z \in A_\tau$,

$$b\Omega \cap R_{C^{-1}\tau}\big(\pi(z)\big) \subset \big\{z' \in b\Omega \;\big|\; z \in R_\tau(z')\big\} \subset b\Omega \cap R_{C\tau}\big(\pi(z)\big).$$

Thus

$$\begin{aligned}\int\limits_{b\Omega} \chi_{R_\tau(z')\cap\Omega}(z)\big(\delta(z',\tau)\big)^{\alpha}\, dS(z') &\lesssim \big(\delta\big(\pi(z),\tau\big)\big)^{\alpha}\mathrm{Area}\big(b\Omega \cap R_{C\tau}\big(\pi(z)\big)\big)\chi_{A_{C\tau}}(z)\\ &\lesssim \tau^2\big(\delta\big(\pi(z),\tau\big)\big)^{\alpha+1/2}\chi_{A_{C\tau}}(z).\end{aligned}$$

The other estimate in Lemma 3.4 follows similarly. □

4. Rescale the $\bar\partial$-Neumann Laplacian

We will keep the notations from the previous section. Let $\widehat{\Omega}_{z'} = \widehat{\Psi}_{z'}(\Omega \cap U_{z'})$ and write

$$\hat{h}(\zeta_1, \operatorname{Im}\zeta_2) = f(\zeta_1) + (\operatorname{Im}\zeta_2)g_1(\zeta_1) + (1/2)(\operatorname{Im}\zeta_2)^2 g_2(\zeta_1) + \sigma_3(\zeta_1, \operatorname{Im}\zeta_2),$$

where $\sigma_3(\zeta_1, \operatorname{Im}\zeta_2) = O(|\operatorname{Im}\zeta_2|^3)$. Then

$$f(\zeta_1) = \widehat{P}(\zeta_1) + O\big(|\zeta_1|^{2m+1}\big), \qquad g_1(\zeta_1) = \widehat{Q}(\zeta_1) + O\big(|\zeta_1|^{m+1}\big), \qquad g_2(\zeta_1) = O\big(|\zeta_1|\big).$$

It is evident that $f \in \widehat{\mathcal{S}}^{0,1}$, $g_1 \in \widehat{\mathcal{S}}^{1,1/2}$, and $g_2 \in \widehat{\mathcal{S}}^{1,0}$.

We flatten the boundary before the rescaling. Let

$$(\eta_1, \eta_2) = \widetilde{\Phi}_{z'}(\zeta_1, \zeta_2) = \big(\zeta_1, \zeta_2 + \hat{h}(\zeta_1, \operatorname{Im}\zeta_2) - F(\zeta_1, \zeta_2)\big),$$

where $F(\zeta_1, \zeta_2) = g_2(\zeta_1)(\operatorname{Re}\zeta_2 + \hat{h}(\zeta_1, \operatorname{Im}\zeta_2))^2/2 + i(g_1(\zeta_1)(\operatorname{Re}\zeta_2) + g_2(\zeta_1)(\operatorname{Re}\zeta_2)(\operatorname{Im}\zeta_2))$. Of course it is not possible to flatten the boundary with a holomorphic change of variables: The term $F(\zeta_1, \zeta_2)$ is added to ensure that $\partial\eta_2/\partial\bar\zeta_2$ vanishes to a desirable higher order at the origin (see the last line in the proof of Lemma 4.2 below). Note that $F \in \widehat{\mathcal{S}}^{1,1}_{z'}$. Let

$$\widetilde{P}_\tau(z') = \big\{|\eta_1| < \tau,\ |\eta_2| < \big(\delta(z', \tau)\big)^{1/2}\big\}.$$

Let $\widetilde{\mathcal{S}}^{a,b}_{z'}$ be the class of smooth functions in the (η_1, η_2)-variables on a neighborhood of the origin defined by (3.12) but with the (ζ_1, ζ_2)-variables replaced by the (η_1, η_2)-variables and $\widehat{P}_\tau(z')$ replaced by $\widetilde{P}_\tau(z')$.

Lemma 4.1. *There exists a constant $C > 0$ such that*

$$\widetilde{P}_{C^{-1}\tau}(z') \subset \widetilde{\Phi}_{z'}\big(\widehat{P}_\tau(z')\big) \subset \widetilde{P}_{C\tau}(z').$$

Proof. The inclusion $\widetilde{\Phi}_{z'}(\widehat{P}_\tau(z')) \subset \widetilde{P}_{C\tau}(z')$ is evident. Now if $\eta \in \widetilde{P}_\tau(z')$, then

$$\begin{aligned} |\zeta_2| &= \big|\eta_2 - \hat{h}(\eta_1, \operatorname{Im}\zeta_2) + F(\eta_1, \zeta_2)\big| \\ &\leqslant \delta^{1/2} + \big|\hat{h}(\eta_1, \operatorname{Im}\zeta_2)\big| + \big|F(\eta_1, \zeta_2)\big| \\ &\lesssim \delta^{1/2} + \tau\delta^{1/2}|\zeta_2| + \tau|\zeta_2|^2 + |\zeta_2|^3. \end{aligned} \tag{4.1}$$

Thus $|\zeta_2| \lesssim \delta^{1/2}$. The other inclusion then follows. $\square$

Let $\check{\rho}(\zeta) = \hat{\rho}(\zeta) - (1/2)g_2(\zeta_1)(\hat{\rho}(\zeta))^2$. Then $\check{\rho}(\zeta)$ is a defining function for $b\widehat{\Omega}_{z'}$ near the origin. Let $\check{r}(z) = \check{\rho}(\widehat{\Psi}_{z'}(z))$. Then $\check{r}(z)$ is a defining function for $b\Omega \cap U_{z'}$ (shrinking $U_{z'}$ if necessary). Let

$$L_1 = \frac{\partial\check{r}}{\partial z_2}\frac{\partial}{\partial z_1} - \frac{\partial\check{r}}{\partial z_1}\frac{\partial}{\partial z_2} \quad \text{and} \quad L_2 = \frac{\partial\check{r}}{\partial\bar{z}_1}\frac{\partial}{\partial z_1} + \frac{\partial\check{r}}{\partial\bar{z}_2}\frac{\partial}{\partial z_2}, \tag{4.2}$$

and let

$$\omega_1 = \frac{\partial \check{r}}{\partial \bar{z}_2} dz_1 - \frac{\partial \check{r}}{\partial \bar{z}_1} dz_2 \quad \text{and} \quad \omega_2 = \frac{\partial \check{r}}{\partial z_1} dz_1 + \frac{\partial \check{r}}{\partial z_2} dz_2. \tag{4.3}$$

Then $\{L_1, L_2\}$ forms an orthogonal basis for $T^{1,0}(\mathbb{C}^2)$ and $\{\omega_1, \omega_2\}$ for $\Lambda^{1,0}(\mathbb{C}^2)$ on $U_{z'}$. Denote by $\widehat{L}_1$, $\widehat{L}_2$, $\widehat{\omega}_1$, and $\widehat{\omega}_2$ the vectors and forms defined as above by replacing $\check{r}$ by $\check{\rho}$, and z_1, z_2 by ζ_1, ζ_2 respectively. Let $\widetilde{L}_k = (\widetilde{\Phi}_{z'})_*(\widehat{L}_k)$, $k = 1, 2$. Write $\zeta_2 = \hat{s} + i\hat{t}$ and $\eta_2 = \tilde{s} + i\tilde{t}$.

Lemma 4.2. *With above notations,*

$$\begin{aligned}
\bar{\widetilde{L}}_1 &= \left(\frac{1}{2} + \alpha_1\right)\frac{\partial}{\partial \bar{\eta}_1} + \left(-\frac{i}{2}\frac{\partial \hat{h}}{\partial \bar{\zeta}_1} + \beta_1\right)\frac{\partial}{\partial \tilde{t}}, \\
\bar{\widetilde{L}}_2 &= \left(\frac{1}{2} + \alpha_2\right)\frac{\partial}{\partial \bar{\eta}_2} + \alpha_3\frac{\partial}{\partial \bar{\eta}_1} + \beta_2\frac{\partial}{\partial \tilde{s}} + \beta_3\frac{\partial}{\partial \tilde{t}},
\end{aligned}$$

where the α*'s are in* $\widetilde{\mathcal{S}}^{0,1/2}_{z'}$ *and the* β*'s in* $\widetilde{\mathcal{S}}^{0,1}_{z'}$.

Proof. By direct computations, we have

$$\begin{aligned}
\bar{\widetilde{L}}_1 &= \frac{\partial \check{\rho}}{\partial \bar{\zeta}_2}\frac{\partial}{\partial \bar{\eta}_1} + \left(\frac{\partial \check{\rho}}{\partial \bar{\zeta}_2}\frac{\partial \tilde{t}}{\partial \bar{\zeta}_1} - \frac{\partial \check{\rho}}{\partial \bar{\zeta}_1}\frac{\partial \tilde{t}}{\partial \bar{\zeta}_2}\right)\frac{\partial}{\partial \tilde{t}}, \\
\bar{\widetilde{L}}_2 &= \left[\frac{\partial \check{\rho}}{\partial \zeta_2}\left(1 + \frac{\partial \hat{h}}{\partial \bar{\zeta}_2} - \frac{\partial \bar{F}}{\partial \bar{\zeta}_2}\right) + \frac{\partial \check{\rho}}{\partial \zeta_1}\left(\frac{\partial \hat{h}}{\partial \bar{\zeta}_1} - \frac{\partial \bar{F}}{\partial \bar{\zeta}_1}\right)\right]\frac{\partial}{\partial \bar{\eta}_2} + \frac{\partial \check{\rho}}{\partial \zeta_1}\frac{\partial}{\partial \bar{\eta}_1} \\
&\quad + \left[\frac{\partial \check{\rho}}{\partial \zeta_1}\left(\frac{\partial \hat{h}}{\partial \bar{\zeta}_1} - \frac{\partial F}{\partial \bar{\zeta}_1}\right) + \frac{\partial \check{\rho}}{\partial \zeta_2}\left(\frac{\partial \hat{h}}{\partial \bar{\zeta}_2} - \frac{\partial F}{\partial \bar{\zeta}_2}\right)\right]\frac{\partial}{\partial \eta_2}.
\end{aligned}$$

Note that $\widetilde{\Phi}^*_{z'}$ is an isomorphism from $\widetilde{\mathcal{S}}^{a,b}_{z'}$ onto $\widehat{\mathcal{S}}^{a,b}_{z'}$. The lemma then follows from the facts that $\hat{h} \in \widehat{\mathcal{S}}^{0,1}_{z'}$, $\check{\rho} - \operatorname{Re}\zeta_2 \in \widehat{\mathcal{S}}^{0,1}_{z'}$, $F \in \widehat{\mathcal{S}}^{1,1}_{z'}$, and

$$\frac{\partial \eta}{\partial \bar{\zeta}_2} = \frac{\partial \hat{h}}{\partial \bar{\zeta}_2} - \frac{\partial F}{\partial \bar{\zeta}_2} = -\frac{i}{2} g_2(\hat{s} + \hat{h})\frac{\partial \hat{h}}{\partial \hat{t}} - \frac{1}{2} g_2 \hat{h} + O\big(|\hat{t}|^2\big) \in \widehat{\mathcal{S}}^{0,1}_{z'}. \qquad \square$$

We now proceed with the rescaling. For any positive τ, let $\delta = \delta(z', \tau)$ be defined by (3.2). Let

$$(w_1, w_2) = D_{z',\tau}(\eta_1, \eta_2) = (\eta_1/\tau, \eta_2/\delta).$$

Let $\widetilde{\Psi}_{z',\tau} = D_{z',\tau} \circ \widetilde{\Phi}_{z'}$ and let $\widetilde{\Omega}_{z',\tau} = \widetilde{\Psi}_{z',\tau}(\widehat{\Omega}_{z'})$. (In what follows, we sometimes suppress the subscript z' for economy of notations when there is no confusion.) Let

$$P_\tau(z') = \big\{|w_1| < 1,\ |w_2| < \delta^{-1/2}\big\}.$$

Let $\mathcal{S}^{a,b}_{z'}$ be the class of functions f depending smoothly on $w \in \mathbb{C}^2$ and $\tau > 0$ such that for any $j, k \geqslant 0$, there exists constants $C_{jk} > 0$, independent of τ, such that

$$\left|D^j_{w_1} D^k_{w_2} f\right| \leqslant C_{jk}\tau^a \delta^{b+k/2}$$

on $P_\tau(z')$ for sufficiently small $\tau > 0$. Here, as before, $D^j_{w_l}$ denotes the partial derivatives of order j with respect to w_l or $\bar{w}_l$. Clearly, if $g \in \widetilde{\mathcal{S}}^{a,b}_{z'}$, then $(D^{-1}_\tau)^*(g) \in \mathcal{S}^{a,b}_{z'}$.

Write $w_1 = x + iy$ and $w_2 = s + it$. Let

$$\begin{aligned}
\overline{\widetilde{L}}_{1,\tau} &= \tau D_{\tau *}(\overline{\widetilde{L}}_1) = \left(\frac{1}{2} + \alpha_1\right)\frac{\partial}{\partial \bar{w}_1} + \left(-\frac{i}{2}\frac{\partial \hat{h}}{\partial \bar{\zeta}_1} + \beta_1\right)\frac{\tau}{\delta}\frac{\partial}{\partial t}, \\
\overline{\widetilde{L}}_{2,\tau} &= \tau D_{\tau *}(\overline{\widetilde{L}}_2) = \left(\frac{1}{2} + \alpha_2\right)\frac{\tau}{\delta}\frac{\partial}{\partial \bar{w}_2} + \alpha_3\frac{\partial}{\partial \bar{w}_1} + \beta_2\frac{\tau}{\delta}\frac{\partial}{\partial s} + \beta_3\frac{\tau}{\delta}\frac{\partial}{\partial t}.
\end{aligned}$$

Write $L^0 = \widetilde{L}_{1,\tau}$ and $L^1 = \overline{\widetilde{L}}_{1,\tau}$. For any tuple $(i_1 \ldots i_l)$ of 0's and 1's, define $L^{(i_1\ldots i_l)}$ inductively by

$$L^{(i_1\ldots i_l)} = \left[L^{i_l}, L^{(i_1\ldots i_{l-1})}\right].$$

Write

$$L^{(i_1\ldots i_l)} = \lambda^{i_1\ldots i_l}\frac{\partial}{\partial t} + a^{i_1\ldots i_l}L^0 + b^{i_1\ldots i_l}L^1.$$

Lemma 4.3. *With the above notations,*

(1) $\lambda^{i_1\ldots i_l} \in \mathcal{S}^{0,0}_{z'}$, $a^{i_1\ldots i_l} \in \mathcal{S}^{1,0}_{z'}$, *and* $b^{i_1\ldots i_l} \in \mathcal{S}^{1,0}_{z'}$.
(2) $(L^{(i_1\ldots i_l)})^* = -L^{((1-i_1)\ldots(1-i_l))} + \sigma$ *for some* $\sigma \in \mathcal{S}^{1,0}_{z'}$.
(3) *There exists a tuple* $(i_1 \ldots i_{l_0})$ *of length* $l_0 \leqslant 2m$ *such that* $|\lambda^{i_1\ldots i_{l_0}}| \gtrsim 1$ *on* $P_\tau(z')$.

Proof. A direction calculation yields that

$$\begin{aligned}
L^{(10)} &= \left(-\frac{i}{4}\left(\frac{\partial}{\partial w_1}\left(\frac{\partial \hat{h}}{\partial \bar{\zeta}_1}\right) + \frac{\partial}{\partial \bar{w}_1}\left(\frac{\partial \hat{h}}{\partial \zeta_1}\right)\right)\frac{\tau}{\delta} + \sigma\right)\frac{\partial}{\partial t} + a^{10}L^0 + b^{10}L^1 \\
&= \left(-\frac{i}{2}\frac{\partial^2 \hat{h}}{\partial \zeta_1 \partial \bar{\zeta}_1}\frac{\tau^2}{\delta} + \sigma\right)\frac{\partial}{\partial t} + a^{10}L^0 + b^{10}L^1
\end{aligned}$$

with $a^{10}, b^{10}, \sigma \in \mathcal{S}^{1,0}_{z'}$. (Here and in what follows, σ could be different in different appearances, but is always in $\mathcal{S}^{1,0}_{z'}$.) It is also easy to see that

$$\left[L^{i_{l+1}}, \frac{\partial}{\partial t}\right] = \sigma\frac{\partial}{\partial t} \quad \mod \left(L^0, L^1\right)$$

where the modulus is with coefficients in $\mathcal{S}^{1,0}_{z'}$. Thus,

$$L^{(i_1\dots i_{l+1})} = \left(L^{i_{l+1}}\lambda^{i_1\dots i_l} + \lambda^{0i_{l+1}}a^{i_1\dots i_l} + \lambda^{1i_{l+1}}b^{i_1\dots i_l} + \sigma\lambda^{i_1\dots i_l}\right)\frac{\partial}{\partial t} \quad \mathrm{mod}\ \left(L^0, L^1\right).$$

Properties (1) and (2) in the lemma then follow from an easy inductive argument on l. To prove (3), one notices from the above formulas that

$$\lambda^{10i_3\dots i_l} = -\frac{i}{2^{l+1}}\frac{\partial^l \hat{h}}{\partial\zeta_1^j\partial\zeta_1^k}\frac{\tau^l}{\delta} + \sigma,$$

where j and k are the numbers of the 0's and 1's in $(10i_3\dots i_l)$ respectively. It follows from the proof of Lemma 3.2 that there exists $j_0, k_0 > 0$ with $j_0 + k_0 = l_0 \leqslant 2m$ such that

$$\left|\frac{\partial^{l_0}\hat{h}}{\partial\zeta_1^{j_0}\partial\bar{\zeta}_1^{k_0}}\frac{\tau^{l_0}}{\delta}\right| \approx 1$$

on $R_\tau(z')$. This then implies the last part of the lemma. □

We define the rescaled $\bar\partial$-Neumann Laplacian through the definition of the associated sesquilinear form. Let $\widehat{\mathcal{G}}_\tau : (L^2(\widetilde{\Omega}_\tau))^2 \to L^2_{(0,1)}(\widehat{\Omega})$ be the transformation defined by

$$\widehat{\mathcal{G}}_\tau(u_1, u_2) = |\det d\widetilde{\Psi}_\tau|^{1/2}\left(u_1(\widetilde{\Psi}_\tau)\bar{\widetilde{\omega}}_1 + u_2(\widetilde{\Psi}_\tau)\bar{\widetilde{\omega}}_2\right),$$

and let $\widehat{\mathcal{G}} : L^2_{(0,1)}(\widehat{\Omega}) \to L^2_{(0,1)}(\Omega\cap U)$ be defined likewise by

$$\widehat{\mathcal{G}}(u_1\bar{\widetilde{\omega}}_1 + u_2\bar{\widetilde{\omega}}_2) = J\widehat{\Psi}(z)\left(u_1(\widehat{\Psi})\bar{\omega}_1 + u_2(\widehat{\Psi})\bar{\omega}_2\right),$$

where $J\Psi$ is the Jacobian determinant of Ψ. Evidently, $\|\widehat{\mathcal{G}}_\tau u\|_{\widehat{\Omega}} \approx \|u\|_{\widetilde{\Omega}_\tau}$ and $\|\widehat{\mathcal{G}}u\|_{\widehat{\Omega}} \approx \|u\|_{\Omega\cap U}$. Let $\mathcal{G}_\tau = \widehat{\mathcal{G}} \circ \widehat{\mathcal{G}}_\tau$ and let $\widetilde{\mathcal{G}}_\tau(u) = (\delta\tau)^{-2}u \circ D_\tau$ be the unitary transformation on L^2-spaces associated with the dilation D_τ. Let

$$Q_\tau(u, v) = \tau^2 Q(\mathcal{G}_\tau u, \mathcal{G}_\tau v) \tag{4.4}$$

be the densely defined, closed sesquilinear form on $(L^2(\widetilde{\Omega}_\tau))^2$ with $\mathrm{Dom}(Q_\tau) = \{\mathcal{G}_\tau^{-1}(u);\ u \in \mathrm{Dom}(Q),\ \mathrm{Supp}\, u \subset \overline{\Omega} \cap U\}$. Here $Q(\cdot,\cdot)$ is the sesquilinear form associated with the $\bar\partial$-Neumann Laplacian on $L^2_{(0,1)}(\Omega)$. The rescaled $\bar\partial$-Neumann Laplacian $\Box_\tau$, formerly written as $\Box_\tau = \tau^2\mathcal{G}_\tau^{-1}\Box\mathcal{G}_\tau$, is the self-adjoint operator associated with Q_τ on forms supported on $P_\tau(z')$ (see the proof of Lemma 6.2).

Lemma 4.4. *For any $u \in \mathrm{Dom}(Q) \cap C_0^\infty(\overline{\Omega} \cap U)$,*

$$Q(u, u) \approx \widehat{Q}\left(\widehat{\mathcal{G}}^{-1}u, \widehat{\mathcal{G}}^{-1}u\right),$$

where $\widehat{Q}$ is the sesquilinear form associated with the $\bar\partial$-Neumann Laplacian on $L^2_{(0,1)}(\widehat{\Omega})$.

Proof. From [38], we know that

$$Q(u,u) \approx \|u\|^2_{\Omega} + \|\bar{L}_1 u\|^2_{\Omega} + \|\bar{L}_2 u\|^2_{\Omega} + \int\limits_{b\Omega} \big(\partial\bar{\partial}\hat{r}(L_1, \bar{L}_1)(z)\big)|u|^2\, dS(z). \tag{4.5}$$

It follows from (A-1) that

$$\begin{aligned} Q(u,u) &\approx \big\|\widehat{\mathcal{G}}^{-1}u\big\|^2_{\widehat{\Omega}} + \big\|\bar{\widehat{L}}_1\widehat{\mathcal{G}}^{-1}u\big\|^2_{\widehat{\Omega}} + \big\|\widehat{\Psi}_*(\bar{L}_2)\widehat{\mathcal{G}}^{-1}u\big\|^2_{\widehat{\Omega}} + \int\limits_{b\widehat{\Omega}} \big(\partial\bar{\partial}\hat{\rho}(\widehat{L}_1, \bar{\widehat{L}}_1)(\zeta)\big)\big|\widehat{\mathcal{G}}^{-1}u\big|^2\, dS(\zeta) \\ &\approx \big\|\widehat{\mathcal{G}}^{-1}u\big\|^2_{\widehat{\Omega}} + \big\|\bar{\widehat{L}}_1\widehat{\mathcal{G}}^{-1}u\big\|^2_{\widehat{\Omega}} + \big\|\bar{\widehat{L}}_2\widehat{\mathcal{G}}^{-1}u\big\|^2_{\widehat{\Omega}} + \int\limits_{b\widehat{\Omega}} \big(\partial\bar{\partial}\hat{\rho}(\widehat{L}_1, \bar{\widehat{L}}_1)(\zeta)\big)\big|\widehat{\mathcal{G}}^{-1}u\big|^2\, dS(\zeta). \end{aligned}$$

Thus $Q(u,u) \approx \widehat{Q}(\widehat{\mathcal{G}}^{-1}u, \widehat{\mathcal{G}}^{-1}u)$. $\square$

Let $\widetilde{u}(\xi', s) = (\mathcal{F}_{\tan} u)(\xi', s)$ be the tangential Fourier transform of u in the $x' = (x, y, t)$ variables. Recall that the tangential Laplacian Λ^ϵ is defined by

$$\mathcal{F}_{\tan}\big(\Lambda^\epsilon u\big)(\xi', s) = \big(1 + |\xi'|^2\big)^{\epsilon/2}\tilde{u}(\xi', s)$$

and the tangential L^2-Sobolev norm of order s by

$$|||u|||^2_\epsilon = \int\limits_{-\infty}^{0}\int\limits_{\mathbb{R}^3} \big(1 + |\xi|^2\big)^\epsilon \big|\tilde{u}(\xi', s)\big|^2\, d\xi'\, ds.$$

Lemma 4.5. *There exists an $\varepsilon \in (0, 1/2]$ such that for any sufficiently small $\tau > 0$,*

$$Q_\tau(u,u) \gtrsim |||u|||^2_\varepsilon + \tau^2\delta^{-2}\left|\left|\left|\frac{\partial u}{\partial \bar{w}_2}\right|\right|\right|^2_{-1+\varepsilon}, \tag{4.6}$$

for all $u \in \mathrm{Dom}(Q_\tau) \cap C_0^\infty(P_\tau(z'))$.

Proof. By Lemma 4.4,

$$\begin{aligned} Q_\tau(u,u) &\approx \tau^2\widehat{Q}(\widehat{\mathcal{G}}_\tau u, \widehat{\mathcal{G}}_\tau u) \\ &\approx \tau^2\big(\|\widehat{\mathcal{G}}_\tau u\|^2_{\widehat{\Omega}} + \|\bar{\widehat{L}}_1\widehat{\mathcal{G}}_\tau u\|^2_{\widehat{\Omega}} + \|\widehat{L}_1\widehat{\mathcal{G}}_\tau u\|^2_{\widehat{\Omega}} + \|\bar{\widehat{L}}_2\widehat{\mathcal{G}}_\tau u\|^2_{\widehat{\Omega}}\big) \\ &\approx \tau^2\|u\|^2_{\widetilde{\Omega}_\tau} + \|\bar{\widetilde{L}}_{1,\tau}u\|^2_{\widetilde{\Omega}_\tau} + \|\widetilde{L}_{1,\tau}u\|^2_{\widetilde{\Omega}_\tau} + \|\bar{\widetilde{L}}_{2,\tau}u\|^2_{\widetilde{\Omega}_\tau}. \end{aligned} \tag{4.7}$$

We first prove that there exists an $\varepsilon \in (0, 1/2]$ such that

$$\left|\left|\left|\frac{\partial u}{\partial t}\right|\right|\right|^2_{-1+\varepsilon} \lesssim Q_\tau(u,u). \tag{4.8}$$

This is a direct consequence of Kohn's method [38], in light of (4.7) and Lemma 4.3. Since we need to keep the track that the constant in (4.8) is independent of τ, we provide the details below.

By Lemma 4.3(1) and (3), we have

$$\left|\!\left|\!\left| \frac{\partial u}{\partial t} \right|\!\right|\!\right|^2_{\varepsilon-1} \lesssim \left|\!\left|\!\left| \lambda^{i_1\dots i_{l_0}} \frac{\partial u}{\partial t} \right|\!\right|\!\right|^2_{\varepsilon-1} \lesssim |\!|\!| L^{(i_1\dots i_{l_0})} u |\!|\!|^2_{\varepsilon-1} + Q_\tau(u,u). \tag{4.9}$$

It remains to estimate $|\!|\!| L^{(i_1\dots i_{l_0})} u |\!|\!|^2_{\varepsilon-1}$, which equals

$$\bigl\langle L^{(i_1\dots i_{l_0-1})} u, \bigl(L^{i_{l_0}}\bigr)^* \Lambda^{2(\varepsilon-1)} L^{(i_1\dots i_{l_0})} u \bigr\rangle - \bigl\langle L^{i_{l_0}} u, \bigl(L^{(i_1\dots i_{l_0-1})}\bigr)^* \Lambda^{2(\varepsilon-1)} L^{(i_1\dots i_{l_0})} u \bigr\rangle. \tag{4.10}$$

The first term above equals

$$\begin{aligned} &\bigl\langle L^{(i_1\dots i_{l_0-1})} u, \Lambda^{2(\varepsilon-1)} L^{(i_1\dots i_{l_0})} \bigl(L^{i_{l_0}}\bigr)^* u \bigr\rangle + \bigl\langle L^{(i_1\dots i_{l_0-1})} u, \bigl[\bigl(L^{i_{l_0}}\bigr)^*, \Lambda^{2(\varepsilon-1)}\bigr] L^{(i_1\dots i_{l_0})} u \bigr\rangle \\ &\quad + \bigl\langle L^{(i_1\dots i_{l_0-1})} u, \Lambda^{2(\varepsilon-1)} \bigl[\bigl(L^{i_{l_0}}\bigr)^*, L^{(i_1\dots i_{l_0})}\bigr] u \bigr\rangle = I + II + III. \end{aligned}$$

We have

$$\begin{aligned} |I| &= \bigl|\bigl\langle \bigl(\Lambda^{2(\varepsilon-1)} L^{(i_1\dots i_{l_0})}\bigr)^* L^{(i_1\dots i_{l_0-1})} u, \bigl(-L^{(1-i_{l_0})} + \sigma\bigr) u \bigr\rangle\bigr| \\ &\lesssim \bigl|\!\bigl|\!\bigl| L^{(i_1\dots i_{l_0-1})} u \bigr|\!\bigr|\!\bigr|^2_{2\varepsilon-1} + Q_\tau(u,u), \end{aligned}$$

because $(\Lambda^{2(\varepsilon-1)} L^{(i_1\dots i_{l_0})})^*$ is a tangential pseudodifferential operator of order $2\varepsilon - 1$. Also, since $[(L^{i_{l_0}})^*, \Lambda^{2(\varepsilon-1)}]$ is of order $2(\varepsilon - 1)$, we have

$$|II| \lesssim C \bigl|\!\bigl|\!\bigl| L^{(i_1\dots i_{l_0-1})} u \bigr|\!\bigr|\!\bigr|^2_{\varepsilon-1} + (1/C) \bigl|\!\bigl|\!\bigl| L^{(i_1\dots i_{l_0})} u \bigr|\!\bigr|\!\bigr|^2_{\varepsilon-1}.$$

Furthermore, as for (4.9), it follows from Lemma 4.3 that

$$\begin{aligned} |III| &= \bigl|\bigl\langle L^{(i_1\dots i_{l_0-1})} u, \Lambda^{2(\varepsilon-1)} \bigl[-L^{(1-i_{l_0})} + \sigma, L^{(i_1\dots i_{l_0})}\bigr] u \bigr\rangle\bigr| \\ &\lesssim C \bigl|\!\bigl|\!\bigl| L^{(i_1\dots i_{l_0-1})} u \bigr|\!\bigr|\!\bigr|^2_{\varepsilon-1} + (1/C) \bigl|\!\bigl|\!\bigl| L^{(i_1\dots i_{l_0})} u \bigr|\!\bigr|\!\bigr|^2_{\varepsilon-1} + Q_\tau(u,u). \end{aligned}$$

The second term in (4.10) is estimated similarly and is left to the reader. From these estimates, we then have

$$\bigl|\!\bigl|\!\bigl| L^{(i_1\dots i_{l_0})} u \bigr|\!\bigr|\!\bigr|^2_{\varepsilon-1} \lesssim \bigl|\!\bigl|\!\bigl| L^{(i_1\dots i_{l_0-1})} u \bigr|\!\bigr|\!\bigr|^2_{2\varepsilon-1} + Q_\tau(u,u).$$

Let $\varepsilon = 2^{-2m}$. Repeating the above arguments, we then obtain (4.8).

Since $\partial/\partial s$ is a linear combination of L^0, L^1, and $\overline{\overline{L}}_2$ with coefficients in $\mathcal{S}^{1,0}_{z'}$, it follows from (4.8) that

$$\left|\!\left|\!\left| \frac{\partial u}{\partial s} \right|\!\right|\!\right|^2_{-1+\varepsilon} \lesssim Q_\tau(u,u). \tag{4.11}$$

Combining (4.7), (4.8), (4.11), and Lemma 4.2, we then have

$$|||\nabla u|||_{\varepsilon-1}^2 + \tau^2\delta^{-2}\left|\left|\left|\frac{\partial u}{\partial \bar{w}_2}\right|\right|\right|_{\varepsilon-1}^2 \lesssim Q_\tau(u,u).$$

By applying the Poincaré inequality to $\tilde{u}(\xi', \cdot)$, we know that the left-hand side above dominates $\|u\|^2$. We thus conclude the proof of the lemma. $\square$

We end this section by remarking that $Q_\tau(u,u)$ is different from the sesquilinear form $Q'_\tau(u,u)$ associated with the $\bar{\partial}$-Neumann Laplacian on the rescaled domain. This is due to, roughly speaking, the fact that anisotropic dilations skew the complex normal direction. More precisely, one has

$$Q_\tau(u,u) \approx Q'_\tau(u,u) + \frac{\tau^2}{\delta^2}\left\|\frac{\partial u}{\partial \bar{w}_2}\right\|^2 \tag{4.12}$$

for forms u supported on $P_\tau(z')$. The last term in (4.6) is a result of the extra term in (4.12). Note that the coefficients of these terms are $\tau^2/\delta^2 \geqslant 1/\delta \to \infty$ as $\tau \to 0$. This fact is essential in the subsequent analysis because it implies that in the limit, one can assume that u is (componentwise) holomorphic with respect to the w_2-variable and hence apply a type of the Paley–Wiener theorem (see Lemma 5.2 below).

5. Auxiliary estimates

Let $\mathbb{C}_- = \{w_2 \in \mathbb{C} \mid \operatorname{Re} w_2 < 0\}$ and let $\mathbb{C}^2_- = \mathbb{C} \times \mathbb{C}_-$. For any ε such that $0 < \varepsilon \leqslant 1/2$ and any $\delta > 0$, let $W_{\varepsilon,\delta}$ be the space of all $u \in L^2(\mathbb{C}^2_-)$ such that

$$\|u\|_{\varepsilon,\delta}^2 = |||u|||_\varepsilon^2 + \delta^{-1}\left|\left|\left|\frac{\partial u}{\partial \bar{w}_2}\right|\right|\right|_{-1+\varepsilon}^2 < \infty. \tag{5.1}$$

Let $\widetilde{Q}_{\varepsilon,\delta}$ be the sesquilinear form on $L^2(\mathbb{C}^2_-)$ associated with the above norm with $\operatorname{Dom}(\widetilde{Q}_{\varepsilon,\delta}) = W_{\varepsilon,\delta}$. Let $\widetilde{\Box}_{\varepsilon,\delta}$ be the associated densely defined, self-adjoint operator on $L^2(\mathbb{C}^2_-)$ and let $\widetilde{N}_{\varepsilon,\delta}$ be its inverse. Let $\chi(w_1, w_2)$ be a smooth cut-off function supported on $\{|w_1| < 1,\ |w_2| < 1\}$ and identically 1 on $\{|w_1| < 1/2,\ |w_2| < 1/2\}$. Let $\chi_\delta(w_1, w_2) = \chi(w_1, \delta^{1/2} w_2)$. We now study the spectral behavior of $\chi_\delta \widetilde{N}_{\varepsilon,\delta}$ as $\delta \to 0^+$. Let P be the orthogonal projection from $L^2(\mathbb{C}^2_-)$ onto $H = \{u \in L^2(\mathbb{C}^2_-) \mid \partial u/\partial \bar{w}_2 = 0\}$. Namely, P is the partial Bergman projection in the w_2-variable on the left-half plane $\mathbb{C}_-$.

Lemma 5.1. *For all $\delta > 0$ and $u \in W_{\varepsilon,\delta}$,*

$$\|(I-P)u\|_{\varepsilon,\delta} \lesssim \|u\|_{\varepsilon,\delta};$$
$$\left|\left|\left|\frac{\partial}{\partial s}(I-P)u\right|\right|\right|_{-1+\varepsilon} + \left|\left|\left|\frac{\partial}{\partial t}(I-P)u\right|\right|\right|_{-1+\varepsilon} \approx \left|\left|\left|\frac{\partial u}{\partial \bar{w}_2}\right|\right|\right|_{-1+\varepsilon}. \tag{5.2}$$

Proof. This lemma follows from standard elliptic theory (compare [47]). Recall that $I - P = 4\frac{\partial}{\partial w_2} G \frac{\partial}{\partial \bar{w}_2}$, where G is the Green's operator in w_2-variable (i.e., the inverse of $-\Delta_{w_2}$). Throughout this section, we will use $\zeta_1 = \xi + i\eta$ and $\zeta_2 = \mu + i\nu$ to denote the dual variables of

$w_1 = x + iy$ and $w_2 = s + it$ in the Fourier transform. Recall that $\tilde{u}$ denotes the tangential Fourier transform of u in the (x, y, t) variables. We have

$$\left|\!\left|\!\left|(I-P)u\right|\!\right|\!\right|_\varepsilon^2 = \int_{-\infty}^{0} ds \int_{\mathbb{R}^3} \left(1+|\zeta_1|^2+\nu^2\right)^\varepsilon \left|2\left(\frac{\partial}{\partial s}+\nu\right)\widetilde{G\left(\frac{\partial u}{\partial \bar{w}_2}\right)}\right|^2 d\xi\, d\eta\, dt.$$

It is easy to check that $\widetilde{G(u)} = -E_+E_-\tilde{u}$, where

$$(E_-\tilde{u})(\zeta_1, s, \nu) = \int_{-\infty}^{s} e^{-|\nu|(s-s')}\tilde{u}(\zeta_1, s', \nu)\, ds'$$

and

$$(E_+\tilde{u})(\zeta_1, s, \nu) = -\int_{s}^{0} e^{|\nu|(s-s')}\tilde{u}(\zeta_1, s', \nu)\, ds'.$$

(See, e.g., Chapter III in [60].) Using the identities $\frac{\partial(E_+\tilde{u})}{\partial s} = |\nu|E_+\tilde{u} + \tilde{u}$ and $E_-(\frac{\partial\tilde{u}}{\partial s}) = -|\nu|E_-(\tilde{u}) + \tilde{u}$, we obtain that

$$2\left(\frac{\partial}{\partial s}+\nu\right)\widetilde{G\left(\frac{\partial u}{\partial \bar{w}_2}\right)} = \left(\nu+|\nu|\right)^2 E_+E_-\tilde{u} - \left(\nu+|\nu|\right)(E_+\tilde{u} - E_-\tilde{u}) - \tilde{u}.$$

Since by the Minkowski inequality,

$$\int_{-\infty}^{0} \left|E\tilde{u}(\zeta_1, s, \nu)\right|^2 ds \leqslant |\nu|^{-2} \int_{-\infty}^{0} \left|\tilde{u}(\zeta_1, s, \nu)\right|^2 ds$$

holds for both E_+ and E_-, we obtain that $|\!|\!|(I-P)u|\!|\!|_\varepsilon \lesssim |\!|\!|u|\!|\!|_\varepsilon$. The first inequality then follows. The second inequality is treated similarly and its proof is left to the reader. □

The following lemma is a simple variation of the classical Paley–Wiener theorem (compare Lemma 4.2 in [47]; see, for example, [59, Chapter III], and [53, Chapter 19], for a treatise of the classical Paley–Wiener theorem). Write $w = w_2 = s + it$.

Lemma 5.2. *Let $m(s)$ be a positive continuous function on $(-\infty, 0]$ such that $\widetilde{m}(\nu) \equiv \int_{-\infty}^{0} e^{2s\nu} m(s)\, ds = \infty$ when $\nu < 0$. Then $\widetilde{\mathcal{R}} : L^2(\mathbb{R}^+, \widetilde{m}(\nu)) \to L^2_h(\mathbb{C}_-, m(s))$ defined by*[3]

$$F(w) = \widetilde{\mathcal{R}}(f)(w) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(\nu) e^{w\nu}\, d\nu \tag{5.3}$$

is unitary and onto.

[3] Here $L^2_h(\mathbb{C}_-, m(s))$ denotes the space of L^2-holomorphic functions on $\mathbb{C}_-$ with respect to the weight $m(s)$.

Proof. A proof is included for the reader's convenience. First, we observe that

$$\widetilde{m}(\nu) = \frac{1}{2\nu} \int_{-\infty}^{0} e^{s'} m\left(\frac{s'}{2\nu}\right) ds' \sim \frac{m(0)}{2\nu},$$

as $\nu \to \infty$. By the Schwarz inequality, we have

$$\int_0^\infty \left|f(\nu)e^{w\nu}\right| d\nu \leqslant \left(\int_0^\infty \left|f(\nu)\right|^2 \widetilde{m}(\nu)\, d\nu\right)^{1/2} \left(\int_0^\infty \frac{e^{2s\nu}}{\widetilde{m}(\nu)}\, d\nu\right)^{1/2} < \infty$$

when $s < 0$. Thus $F(w)$ defined by (5.2) is holomorphic on $\mathbb{C}_-$. That $\mathcal{R}$ is unitary follows easily from the Plancherel and Fubini theorems. It remains to prove that $\mathcal{R}$ is onto. Now let $F(w) \in L^2_h(\mathbb{C}_-, m(s))$. Then there exists a sequence of positive numbers $\alpha_j \to \infty$ such that

$$\int_{-\infty}^{0} \left|F(s \pm \alpha_j i)\right|^2 m(s)\, ds \to 0. \tag{5.4}$$

Define

$$f(\nu) = \lim_{\alpha_j \to \infty} \frac{e^{-s\nu}}{\sqrt{2\pi}} \int_{-\alpha_j}^{\alpha_j} F(s+it)e^{-it\nu}\, dt$$

where the convergence is in L^2-norm. It follows from Cauchy's theorem and (5.4) that the above definition of $f(\nu)$ is independent of s. By the Plancherel and Fubini theorems again, we have

$$\int_{-\infty}^{\infty} \left|f(\nu)\right|^2 \widetilde{m}(\nu)\, d\nu = \|F\|^2 < \infty.$$

Since $\widetilde{m}(\nu) = \infty$ for $\nu < 0$, we have $f(\nu) = 0$ a.e. on $(-\infty, 0)$. $\square$

Lemma 5.3. *For sufficiently small $\delta > 0$ and sufficiently large j,*

$$s_j\left(\chi_\delta \widetilde{N}^{1/2}_{\varepsilon,\delta}\right) \lesssim \left(1 + j\delta^{1/2}\right)^{-\varepsilon/4}.$$

Proof. For $u \in L^2(\mathbb{C}^+_-)$, we write $\widetilde{N}^{1/2}_{\varepsilon,\delta} u = (I - P)\widetilde{N}^{1/2}_{\varepsilon,\delta} u + P\widetilde{N}^{1/2}_{\varepsilon,\delta} u = v_1 + v_2$. We first study $\chi_\delta(I - P)\widetilde{N}_{\varepsilon,\delta}$. We extend v_1 evenly to $s > 0$ by letting $v_1(w_1, s+it) = v_1(w_1, -s+it)$. Denote by $\widehat{v_1}$ the Fourier transform of the extended v_1 in all variables. Then by Lemma 5.1,

$$\|\widetilde{N}_{\varepsilon,\delta}^{1/2}u\|_{\varepsilon,\delta}^2 \gtrsim \int_{\mathbb{R}^4} \left(\left(1+|\zeta_1|^2+\nu^2\right)^{\varepsilon} + \delta^{-1}|\zeta_2|^2\left(1+|\zeta_1|^2+\nu^2\right)^{-1+\varepsilon}\right)|\widehat{v}_1|^2\,dV(\zeta)$$
$$\gtrsim \int_{\mathbb{R}^4} \left(1+|\zeta_1|^2+\delta^{-1}|\zeta_2|^2\right)^{\varepsilon}|\widehat{v}_1|^2\,dV(\zeta) \equiv \|v_1\|_{\varepsilon,\delta}^{\prime 2}, \tag{5.5}$$

where in the last estimate we use the following simple inequality: $a^{\varepsilon}b^{1-\varepsilon} \leqslant \varepsilon a + (1-\varepsilon)b$. Let $\Delta'_{\varepsilon,\delta}$ be the Dirichlet realization of the self-adjoint operator associated with the sesquilinear form that defines the norm $\|\cdot\|'_{\varepsilon,\delta}$ on $\{|w_1|<1,\ |w_2|<\delta^{-1/2}\}$. As such, we have $\|v\|'_{\varepsilon,\delta} = \|(\Delta'_{\varepsilon,\delta})^{1/2}v\|$. Let $S_\delta u(w_1,w_2) = \delta^{1/2}u(w_1,\delta^{1/2}w_2)$. Then S_δ is an isometry on $L^2(\mathbb{C}^2_-)$. Furthermore, it is easy to see that $\|v\|'_{\varepsilon,\delta} = \|S_\delta v\|'_{\varepsilon,1}$. It follows that

$$\lambda_j\left(\Delta'_{\varepsilon,\delta}\right) = \lambda_j\left(\Delta'_{\varepsilon,1}\right) \approx \lambda_j\left(\Delta^{\varepsilon}\right) \approx j^{\varepsilon/2}, \tag{5.6}$$

where Δ is the usual Dirichlet Laplacian on $\{|w_1|<1, |w_2|<1\}$ and the last estimate follows from the classical Weyl formula. Thus it follows from (5.5) that

$$\left\|\left(\Delta'_{\varepsilon,\delta}\right)^{1/2}\chi_\delta(I-P)\widetilde{N}_{\varepsilon,\delta}u\right\|^2 = \|\chi_\delta v_1\|_{\varepsilon,\delta}^{\prime 2} \lesssim \|v_1\|_{\varepsilon,\delta}^{\prime 2} \lesssim \left\|\widetilde{N}_{\varepsilon,\delta}^{1/2}u\right\|_{\varepsilon,\delta}^2 = \|u\|^2.$$

Therefore, by (2.3) and (5.6), we have

$$s_j\left(\chi_\delta(I-P)\widetilde{N}_{\varepsilon,\delta}\right) \lesssim s_j\left(\left(\Delta'_{\varepsilon,\delta}\right)^{-1/2}\right) \approx (1+j)^{-\varepsilon/4}. \tag{5.7}$$

We now study the eigenvalues of $\chi_\delta P\widetilde{N}_{\varepsilon,\delta}$. Let $\mathcal{R}: L^2(\mathbb{C}\times(0,\infty)) \to H$ be defined by

$$\mathcal{R}\phi(w_1,w_2) = \frac{1}{\sqrt{\pi}}\int_0^\infty e^{w_2\nu}\phi(w_1,\nu)\sqrt{\nu}\,d\nu,$$

and let $\mathcal{R}^*: L^2(\mathbb{C}^2_-) \to L^2(\mathbb{C}\times(0,\infty))$ be defined by

$$\mathcal{R}^*u(w_1,\nu) = \sqrt{\frac{\nu}{\pi}}\int_{-\infty}^0 (\mathcal{F}_t u)(w_1,s,\nu)e^{s\nu}\,ds,$$

where, as usual, $\mathcal{F}_t$ is the Fourier transform in the t-variable. It follows from Lemma 5.2 (with $m(s)=1$) that $\mathcal{R}$ is unitary and onto. Furthermore, $\mathcal{R}^*\mathcal{R} = I$ and $\mathcal{R}\mathcal{R}^* = P$.

For any $\lambda > 1$, let $\mathcal{E}_\lambda : L^2(\mathbb{C}\times(0,\infty)) \to L^2(\mathbb{C}\times(0,\infty))$ be defined by

$$\mathcal{E}_\lambda\phi(w_1,\nu) = \left(\mathcal{F}_{w_1}^{-1}\chi_{\{1+|\zeta_1|^2+\nu^2<\lambda^2\}}\mathcal{F}_{w_1}\phi\right)(w_1,\nu),$$

where χ_A is the characteristic function for the set A as before and $\mathcal{F}_{w_1}$ is the Fourier transform in the x and y variables. (Recall that $w_1 = x+iy$, $w_2 = s+it$, and their dual variables are $\zeta_1 = \xi + i\eta$ and $\zeta_2 = \mu + i\nu$.) Let $\mathcal{M}_\lambda = R\mathcal{E}_\lambda R^* : L^2(\mathbb{C}^2_-) \to H$. Then $\mathcal{M}_\lambda$ is an orthogonal project into H. A straightforward calculation yields that the kernel M_λ of $\mathcal{M}_\lambda$ is given by

$$M_\lambda(w,w')=\frac{1}{4\pi^3}\int_0^\infty\int_{\mathbb{C}} \nu e^{(w_2+\overline{w}'_2)\nu+i((x-x')\xi+(y-y')\eta)}\chi_{\{1+|\zeta_1|^2+\nu^2<\lambda^2\}}\,d\xi\,d\eta\,d\nu.$$

Thus,

$$M_\lambda(w,w)=\frac{1}{4\pi^2}\int_0^{\sqrt{\lambda^2-1}} \nu e^{2s\nu}\big(\lambda^2-1-\nu^2\big)\,d\nu.$$

The square of the Hilbert–Schmidt norm of $\chi_\delta\mathcal{M}_\lambda$ equals

$$\begin{aligned}\int_{\mathbb{C}^4}\big|\chi_\delta(w)M_\lambda(w,w')\big|^2\,dV(w,w') &= \int_{\mathbb{C}^2}|\chi_\delta|^2M_\lambda(w,w)\,dV(w)\\ &=\frac{1}{4\pi^2\delta}\int_{\mathbb{C}^2}\big|\chi(w_1,w_2)\big|^2M_\lambda\big(w_1,\delta^{-1/2}w_2\big)\,dV(w)\\ &\lesssim \delta^{-1}\int_{-\infty}^0 ds\int_0^{\sqrt{\lambda^2-1}} \nu e^{2\delta^{-1/2}s\nu}\big(\lambda^2-1-\nu^2\big)\,d\nu\\ &\lesssim \delta^{-1/2}\int_0^{\sqrt{\lambda^2-1}}\big(\lambda^2-1-\nu^2\big)\,d\nu\lesssim\delta^{-1/2}\lambda^3.\end{aligned}$$

Thus, on the one hand, we have

$$s_j(\chi_\delta\mathcal{M}_\lambda)\lesssim\big(\delta^{-1/2}\lambda^3/j\big)^{1/2}. \tag{5.8}$$

On the other hand, we have

$$\begin{aligned}\|v_2-\mathcal{M}_\lambda v_2\|^2&=\big\|\mathcal{R}\mathcal{R}^*v_2-\mathcal{R}\mathcal{E}_\lambda\mathcal{R}^*v_2\big\|^2=\big\|(I-\mathcal{E}_\lambda)\mathcal{R}^*v_2\big\|^2\\ &=\big\|(1-\chi_{\{1+|\zeta_1|^2+\nu^2<\lambda^2\}})\mathcal{F}_{w_1}\mathcal{R}^*v_2\big\|^2\\ &\lesssim\int_0^\infty d\nu\int_{\mathbb{C}}\chi_{\{1+|\zeta_1|^2+\nu^2\geqslant\lambda^2\}}\,d\xi\,d\eta\left|\sqrt{\nu}\int_{-\infty}^0(\mathcal{F}_{w_1}\mathcal{F}_tv_2)(\zeta_1,\nu,s)e^{s\nu}\,ds\right|^2\\ &\lesssim\int_0^\infty d\nu\int_{\mathbb{C}}\chi_{\{1+|\zeta_1|^2+\nu^2\geqslant\lambda^2\}}\,d\xi\,d\eta\int_{-\infty}^0|\mathcal{F}_{w_1}\mathcal{F}_tv_2|^2\,ds\\ &\lesssim\lambda^{-2\varepsilon}|||v_2|||_\varepsilon^2\lesssim\lambda^{-2\varepsilon}\big\|\widetilde{N}_{\varepsilon,\delta}^{1/2}u\big\|_{\varepsilon,\delta}^2=\lambda^{-2\varepsilon}\|u\|^2.\end{aligned}\tag{5.9}$$

(Here we have used Lemma 5.1 in the last estimate.) From (5.8), (5.9), and (2.1), we obtain

$$s_j\big(\chi_\delta P\widetilde{N}^{1/2}_{\varepsilon,\delta}\big) \lesssim \big(\delta^{-1/2}\lambda^3 j^{-1}\big)^{1/2} + \lambda^{-\varepsilon}$$
$$\lesssim \big(j\delta^{1/2}\big)^{-1/8} + \big(j\delta^{1/2}\big)^{-\varepsilon/4} \lesssim \big(j\delta^{1/2}\big)^{-\varepsilon/4}, \tag{5.10}$$

by taking $\lambda = (j\delta^{1/2})^{1/4}$. This estimate, combined with (5.7), then gives us the desired estimate. $\square$

6. Estimates on the spectral kernel

Let $E(\lambda)$ be the spectral resolution of the $\bar\partial$-Neumann Laplacian $\square$ on $L^2_{(0,1)}(\Omega)$ and let $e(\lambda; z, \zeta)$ be its kernel (in the sense of Schwartz). It follows from the finite type and pseudoconvexity assumptions on Ω that $\square$ is subelliptic [38] and hence has purely discrete spectrum. Let λ_j be the eigenvalues of $\square$. Let φ_j be the normalized associated eigenforms. Then the spectral resolution $E(\lambda)$ is given by

$$E(\lambda) f = \sum_{\lambda_j \leqslant \lambda} \langle f, \varphi_j\rangle \varphi_j,$$

and the spectral kernel $e(\lambda; z, \zeta)$ can be written as

$$e(\lambda; z, \zeta) = \sum_{\lambda_j \leqslant \lambda} \varphi_j(z) \wedge \overline{\varphi_j(\zeta)}.$$

Note that

$$\operatorname{tr} e(\lambda; z, z) = \sum_{\lambda_j \leqslant \lambda} \big|\varphi_j(z)\big|^2.$$

By the classical elliptic theory we know that

$$\lim_{\lambda\to\infty} \lambda^{-2} \operatorname{tr} e(\lambda; z, z) = (2\pi)^{-2}, \tag{6.1}$$

where the limit is uniform on any compact subset of Ω (e.g., [30]). In fact, for any $C > 0$, $\operatorname{tr} e(\lambda; z, z) \lesssim \lambda^2$ for all $z \in \Omega$ with $d(z) \geqslant C\lambda^{-1/2}$ (e.g., [47]).

Lemma 6.1. *Let $\tau = 1/\sqrt{\lambda}$. For any $C > 0$,*

$$\operatorname{tr} e(\lambda; z, z) \lesssim \lambda\big(\delta\big(\pi(z), \tau\big)\big)^{-1}, \tag{6.2}$$

for sufficiently large $\lambda > 0$ and all $z \in \Omega$ with $d(z) \geqslant C(\delta(\pi(z), \tau))^{1/2}$.

Proof. In light of the above remarks, it suffices to prove (6.2) when $C(\delta(\pi(z), \tau))^{1/2} \leqslant d(z) \leqslant \lambda^{-1/2}$. We will use a global rescaling scheme which is slightly different from the local rescaling scheme introduced in Section 4.

Let $z' \in b\Omega$ and let $\Omega' = \Phi'(\Omega)$ as in Section 3 where Φ' is given by (3.4). Let $\delta = \delta(z', \tau)$. For any σ such that $C\sqrt{\delta} \leqslant \sigma \leqslant \tau$, let $(\xi_1', \xi_2') = D'_{\tau,\sigma}(\xi_1, \xi_2) = (\xi_1/\tau, \xi_2/\sigma)$. Let $\Omega'_{\tau,\sigma} = D'_{\tau,\sigma}(\Omega)$. Let

$$L_1' = \frac{1}{|\partial r|}\left(\frac{\partial r}{\partial z_2}\frac{\partial}{\partial z_1} - \frac{\partial r}{\partial z_1}\frac{\partial}{\partial z_2}\right), \qquad L_2' = \frac{1}{|\partial r|}\left(\frac{\partial r}{\partial \bar z_1}\frac{\partial}{\partial z_1} + \frac{\partial r}{\partial \bar z_2}\frac{\partial}{\partial z_2}\right). \tag{6.3}$$

We extend the vector fields L_1' and L_2' to form an orthonormal basis for $T^{1,0}(\mathbb{C}^2)$ over $\overline{\Omega}$. Let ω_1' and ω_2' be the dual basis. Denote by $L_1'^{\tau,\sigma}$, $L_2'^{\tau,\sigma}$, $\omega_1'^{\tau,\sigma}$, and $\omega_1'^{\tau,\sigma}$ the vector fields and forms defined as above but with z replaced by ξ' and r replaced by $\rho_{\tau,\sigma}(\xi') = (1/\sigma)\rho(\tau\xi_1', \sigma\xi_2')$, where ρ is given by (3.5). Let $\mathcal{H}_{\tau,\sigma} : L^2_{0,1}(\Omega'_{\tau,\sigma}) \to L^2_{0,1}(\Omega)$ be the unitary transformation defined by

$$\mathcal{H}_{\tau,\sigma}\big(v_1\bar\omega_1'^{\tau,\sigma} + v_2\bar\omega_2'^{\tau,\sigma}\big) = (\tau\sigma)^{-1} J\Phi'(z)\big(v_1\big(D'_{\tau,\sigma}\circ\Phi'\big)\bar\omega_1' + v_2\big(D'_{\tau,\sigma}\circ\Phi'\big)\bar\omega_2'\big),$$

where as before $J\Phi'$ is the Jacobian determinant of Φ'. Let

$$Q^*_{\tau,\sigma}(u,v) = \tau^2 Q(\mathcal{H}_{\tau,\sigma}u, \mathcal{H}_{\tau,\sigma}v)$$

with $\mathrm{Dom}(Q^*_{\tau,\sigma}) = \mathrm{Dom}(Q'_{\tau,\sigma})$, where Q is the sesquilinear form associated with the $\bar\partial$-Neumann Laplacian $\Box$ on Ω as before, and $Q'_{\tau,\sigma}$ associated with the $\bar\partial$-Neumann Laplacian $\Box'_{\tau,\sigma}$ on $\Omega'_{\tau,\sigma}$. Let $\Box^*_{\tau,\sigma}$ be the self-adjoint operator associated with $Q^*_{\tau,\sigma}$ and let $e^*_{\tau,\sigma}(\lambda;\xi',\eta')$ be the kernel of the spectral resolution of $\Box^*_{\tau,\sigma}$. Then

$$e(\lambda;z,\zeta) = (\tau\sigma)^{-2} J\Phi'(z)\overline{J\Phi'(\zeta)} e^*_{\tau,\sigma}\big(\tau^2\lambda; D'_{\tau,\sigma}\circ\Phi'(z), D'_{\tau,\sigma}\circ\Phi'(\zeta)\big). \tag{6.4}$$

Let $P' = \{\xi' \in \mathbb{C}^2 \mid |\xi_1'| < c,\ |\xi_2' + 1/2| < c\}$. Then for sufficiently small $c > 0$, P' is relatively compact subset of $\Omega'_{\tau,\sigma}$. Furthermore, if u is supported in P', then

$$Q^*_{\tau,\sigma}(u,u) \gtrsim \tau^2\big\|(\tau\sigma)^{-1}\nabla_\xi u\big(\tau^{-1}\xi_1, \sigma^{-1}\xi_2\big)\big\|^2_{\Omega'} \gtrsim \|\nabla_{\xi'}u\|^2_{\Omega'_{\tau,\sigma}}, \tag{6.5}$$

where the rescaling of u is component-wise and last estimate follows from $\sigma \lesssim \tau$. Let $z = (\Phi')^{-1}(0, -\sigma/2)$. Then $d(z) \approx \sigma$. From (6.4), we have

$$\operatorname{tr} e(\lambda;\ z,z) \lesssim (\tau\sigma)^{-2} \operatorname{tr} e^*_{\tau,\sigma}\big(1; (0,-1/2), (0,-1/2)\big). \tag{6.6}$$

By (6.5), the Sobolev lemma and Gårding inequality, we have for any $k > 2$ and $\xi' \in P'$

$$\begin{aligned}\big\|\operatorname{tr} e^*_{\tau,\sigma}\big(\tau^2\lambda;\cdot,\xi'\big)\big\|_{L^\infty(P')} &\lesssim \big\|\Box^{*k}_{\tau,\sigma}\operatorname{tr} e^*_{\tau,\sigma}\big(\tau^2\lambda;\cdot,\xi'\big)\big\|_{L^2(\Omega'_{\tau,\sigma})} + \big\|\operatorname{tr} e^*_{\tau,\sigma}\big(\tau^2\lambda;\cdot,\xi'\big)\big\|_{L^2(\Omega'_{\tau,\sigma})} \\ &\lesssim \big(1 + \big(\tau^2\lambda\big)^k\big)\big(\operatorname{tr} e^*_{\tau,\sigma}\big(\tau^2\lambda;\xi',\xi'\big)\big)^{1/2}.\end{aligned}$$

Therefore $\operatorname{tr} e^*_{\tau,\sigma}(1;(0,-1/2),(0,-1/2)) \lesssim 1$. As z' varies on $b\Omega$ and σ varies between $\sqrt{\delta} \lesssim \sigma \lesssim \tau$, we obtain (6.2) from (6.6) for $z \in \Omega$ with $\delta^{1/2} \lesssim d(z) \lesssim \tau$. This concludes the proof of the lemma. □

We now estimate the spectral kernel $e(\lambda;z,z)$ for z close to the boundary using the rescaling scheme in Section 4. We will keep the notations of Section 4. Let κ be a cut-off function compactly supported on $U_{z'}$ and identically 1 on a neighborhood of z' of uniform size. Let

$$E_\tau(\lambda) = \mathcal{G}_\tau^{-1}\kappa E\big(\lambda/\tau^2\big)\kappa\mathcal{G}_\tau : \big(L^2(\widetilde\Omega_\tau)\big)^2 \to \big(L^2(\widetilde\Omega_\tau)\big)^2.$$

Then the kernel of $E_\tau(\lambda)$ is given by

$$e_\tau(\lambda; w, w') = e\big(\lambda/\tau^2; \Psi_\tau^{-1}(w), \Psi_\tau^{-1}(w')\big)\kappa(w)\kappa(w') \\ \times \left|\det d\Psi_\tau^{-1}(w)\right|^{\frac{1}{2}} \left|\det d\Psi_\tau^{-1}(w')\right|^{\frac{1}{2}}, \tag{6.7}$$

where $\Psi_\tau = \widetilde{\Psi}_\tau \circ \widehat{\Psi}$.

Lemma 6.2. *Let $\tau = 1/\sqrt{\lambda}$. Then for any $z' \in b\Omega$ and sufficiently large $\lambda > 0$,*

$$\int_{R_\tau(z')\cap\Omega} \operatorname{tr} e(\lambda; z, z)\, dV(z) \lesssim \big(\delta(z', \tau)\big)^{-1/2}. \tag{6.8}$$

Proof. In light of (6.7), it suffices to prove

$$\int_{P_\tau(z')\cap\widetilde{\Omega}_\tau} \operatorname{tr} e_\tau(1; w, w)\, dV(w) \lesssim \big(\delta(z', \tau)\big)^{-1/2}. \tag{6.9}$$

Let $\Box_\tau : (L^2(\widetilde{\Omega}_\tau \cap P_\tau(z')))^2 \to (L^2(\widetilde{\Omega}_\tau \cap P_\tau(z')))^2$ be the operator associated with the sesquilinear form Q_τ that is given by (4.4) and has domain $\{\mathcal{G}_\tau^{-1}(u) \mid u \in \operatorname{Dom}(Q), \operatorname{Supp} u \subset \overline{\Omega} \cap \Psi_\tau^{-1}(P_\tau(z'))\}$. Note that $\Box_\tau = \tau^2 \mathcal{G}_\tau^{-1} \Box \mathcal{G}_\tau$. Also by Lemma 4.1, $R_{C^{-1}\tau}(z') \subset \Psi_\tau^{-1}(P_\tau(z')) \subset R_{C\tau}(z')$ for some sufficiently large constant $C > 0$. By Lemma 4.5 and using the fact that $\delta = \delta(z', \tau) \lesssim \tau^2$, we have

$$Q_\tau(u, u) \gtrsim \|u\|^2_{\varepsilon, C\delta},$$

for $u \in \operatorname{Dom}(\Box_\tau^{1/2})$, where $C > 0$ is any constant. Therefore,

$$\|u\|^2 = Q_\tau\big(N_\tau^{1/2}u, N_\tau^{1/2}u\big) \gtrsim \big\|\widetilde{\Box}^{1/2}_{\varepsilon, C\delta} N_\tau^{1/2} u\big\|^2.$$

Choose $C > 2$. Then $N_\tau^{1/2} = \chi_{C\delta} N_\tau^{1/2} = \chi_{C\delta} \widetilde{N}^{1/2}_{\varepsilon, C\delta} \widetilde{\Box}^{1/2}_{\varepsilon, C\delta} N_\tau^{1/2}$. It follows from (2.3) and Lemma 5.3 that

$$s_j\big(N_\tau^{1/2}\big) \lesssim s_j\big(\chi_{C\delta} \widetilde{N}^{1/2}_{\varepsilon, \delta}\big) \lesssim \big(j\delta^{1/2}\big)^{-\varepsilon/4}. \tag{6.10}$$

Let K be any positive integer such that $K > 4/\varepsilon$. Let $\chi^{(k)}$, $k = 0, 1, \ldots, K$, be a family of cut-off functions supported in $\{|w_1| < 1, |w_2| < 1\}$ such that $\chi^{(0)} = \chi$ and $\chi^{(k+1)} = 1$ on $\operatorname{Supp} \chi^{(k)}$. Let

$$E_\tau^{(l)}(\lambda) = \mathcal{G}_\tau^{-1} \kappa \big(\tau^2 \Box\big)^l E\big(\lambda \tau^{-2}\big) \kappa \mathcal{G}_\tau : \big(L^2(\widetilde{\Omega}_\tau)\big)^2 \to \big(L^2(\widetilde{\Omega}_\tau)\big)^2.$$

Note that $E_\tau^{(0)}(\lambda) = E_\tau(\lambda)$ and

$$\big\| E_\tau^{(l)}(1) u \big\| \lesssim \|u\|. \tag{6.11}$$

It is easy to check the following commutating identity:

$$Q(\theta u, \theta u) = \mathrm{Re}(\theta\Box u, \theta u) + (1/2)\big(u, [\theta, A]u\big) \tag{6.12}$$

where θ is any smooth function and $A = [\bar\partial^*, \theta]\bar\partial + \bar\partial[\bar\partial^*, \theta] + \bar\partial^*[\bar\partial, \theta] + [\bar\partial, \theta]\bar\partial^*$. Note that $[\theta, A]$ is of zero order. Using the above identity and the Schwarz inequality, we obtain that for any $u \in (L^2(\widetilde{\Omega}_\tau))^2$,

$$\begin{aligned}
&\big\|\Box_\tau^{\frac{1}{2}}\chi_\delta^{(k)}E_\tau^{(l)}(1)u\big\|^2 \\
&\quad= Q_\tau\big(\chi_\delta^{(k)}E_\tau^{(l)}(1)u,\ \chi_\delta^{(k)}E_\tau^{(l)}(1)u\big) \\
&\quad= \tau^2 Q\big(\chi_\delta^{(k)}(\Psi_\tau)\big(\tau^2\Box\big)^l E\big(\tau^{-2}\big)\kappa\mathcal{G}_\tau u, \chi_\delta^{(k)}(\Psi_\tau)\big(\tau^2\Box\big)^l E\big(\tau^{-2}\big)\kappa\mathcal{G}_\tau u\big) \\
&\quad\lesssim \big\|\chi_\delta^{(k)}(\Psi_\tau)\big(\tau^2\Box\big)^{l+1}E\big(\tau^{-2}\big)\kappa\mathcal{G}_\tau u\big\|^2 + \big\|\chi_\delta^{(k+1)}(\Psi_\tau)\big(\tau^2\Box\big)^l E\big(\tau^{-2}\big)\kappa\mathcal{G}_\tau u\big\|^2 \\
&\quad= \big\|\chi_\delta^{(k)}E_\tau^{(l+1)}(1)u\big\|^2 + \big\|\chi_\delta^{(k+1)}E_\tau^{(l)}(1)u\big\|^2.
\end{aligned}$$

By (2.2) and (2.3), we have

$$\begin{aligned}
s_{3j+1}\big(\chi_\delta^{(k)}E_\tau^{(l)}(1)\big) &\leqslant s_{j+1}\big(N_\tau^{1/2}\big)s_{2j+1}\big(\Box_\tau^{1/2}\chi_\delta^{(k)}E_\tau^{(l)}(1)\big) \\
&\leqslant s_{j+1}\big(N_\tau^{1/2}\big)\big(s_{j+1}\big(\chi_\delta^{(k)}E_\tau^{(l+1)}(1)\big) + s_{j+1}\big(\chi_\delta^{(k+1)}E_\tau^{(l)}(1)\big)\big).
\end{aligned} \tag{6.13}$$

Using (6.10), (6.11), and (6.13), we then obtain by an inductive argument on $K - (k+l)$ that

$$s_j\big(\chi_\delta^{(k)}E_\tau^{(l)}(1)\big) \lesssim \big(j\delta^{1/2}\big)^{-(K-(k+l))\varepsilon/4} \tag{6.14}$$

for any pair of non-negative integers k, l such that $0 \leqslant k + l \leqslant K$. In particular,

$$s_j\big(\chi_\delta E_\tau(1)\big) \lesssim \big(j\delta^{1/2}\big)^{-K\varepsilon/4}.$$

Since $E_\tau(1)$ is a contraction, we also have that $s_j(\chi_\delta E_\tau(1)) \lesssim 1$. The trace norm of $\chi_\delta E_\tau(1)$ is then given by

$$\sum_{j\leqslant\delta^{-1/2}} s_j\big(\chi_\delta E_\tau(1)\big) + \sum_{j>\delta^{-1/2}} s_j\big(\chi_\delta E_\tau(1)\big) \lesssim \delta^{-1/2} + \sum_{j>\delta^{-1/2}} \big(j\delta^{-1/2}\big)^{-K\varepsilon/4} \lesssim \delta^{-1/2}.$$

Inequality (6.9) is now an easy consequence of the above estimate. $\Box$

Lemma 6.3. *For sufficiently large $\lambda > 0$,*

$$\int_{A_\tau} \operatorname{tr} e(\lambda; z, z)\, dV(z) \lesssim \tau^{-2}\int_{b\Omega} \big(\delta(z', \tau)\big)^{-1}\, dS(z') \lesssim \lambda^{1+m}. \tag{6.15}$$

Proof. By Lemma 6.2, we have

$$\big(\delta(z',\tau)\big)^{-1/2}\int_{\Omega}\chi_{\Omega\cap R_\tau(z')}(z)\,\mathrm{tr}\,e(\lambda;z,z)\,dV(z)\lesssim\big(\delta(z',\tau)\big)^{-1}.$$

Integrating both sides with respect to $z'\in b\Omega$ and using the Fubini–Tonelli Theorem, we have

$$\int_{\Omega}\mathrm{tr}\,e(\lambda;z,z)\,dV(z)\int_{z'\in b\Omega}\chi_{\Omega\cap R_\tau(z')}(z)\big(\delta(z',\tau)\big)^{-1/2}\,dS(z')\lesssim\int_{b\Omega}\big(\delta(z',\tau)\big)^{-1}\,dS(z').$$

By Lemma 3.4 and the fact that $\delta(z',\tau)\gtrsim\tau^{2m}$, we then obtain

$$\int_{A_{C\tau}}\mathrm{tr}\,e(\lambda;z,z)\,dV(z)\lesssim\tau^{-2}\int_{b\Omega}\big(\delta(z',\tau)\big)^{-1}\,dS(z')\lesssim\lambda^{1+m}.$$

Lemma 6.3 then follows from a rescaling of τ in the above arguments. □

We are now in a position to prove Theorem 1.2. Note that

$$\begin{aligned}\frac{\mathcal{N}(\lambda)}{\lambda^{1+m}}&=\lambda^{-1-m}\int_{\Omega}\mathrm{tr}\,e(\lambda;z,z)\,dV(z)\\&=\lambda^{-1-m}\int_{A_\tau}\mathrm{tr}\,e(\lambda;z,z)\,dV(z)+\lambda^{-1-m}\int_{\Omega\setminus A_\tau}\mathrm{tr}\,e(\lambda;z,z)\,dV(z).\end{aligned}$$

By Lemma 6.3, the first term in the last expression is bounded from above by a constant multiple of

$$\lambda^{-m}\int_{b\Omega}\big(\delta(z',1/\sqrt{\lambda})\big)^{-1}\,dV(z).$$

Since $\lim_{\lambda\to\infty}\lambda^{-m}(\delta(z',1/\sqrt{\lambda}))^{-1}=0$ except when z' is of type $2m$ and the boundary points with type $2m$ $(m>1)$ has zero Lebesgue measure, we have

$$\lim_{\lambda\to\infty}\lambda^{-m}\int_{b\Omega}\big(\delta(z',1/\sqrt{\lambda})\big)^{-1}\,dS(z')=0.$$

By (6.1), Lemma 6.1, and the Lebesgue dominated convergence theorem, we have

$$\lim_{\lambda\to\infty}\lambda^{-1-m}\int_{\Omega\setminus A_\tau}\mathrm{tr}\,e(\lambda;z,z)\,dV(z)=\begin{cases}(2\pi)^{-2}\,\mathrm{vol}(\Omega), & \text{if } m=1,\\ 0, & \text{if } m>1.\end{cases}$$

We then conclude the proof of Theorem 1.2.

Remarks. It follows from [47] that when $m = 1$,

$$\lim_{\lambda\to\infty} \frac{\mathcal{N}(\lambda)}{\lambda^2} = \frac{|\Omega|}{(2\pi)^2} + \frac{1}{96\pi^2} \int\limits_{b\Omega} \frac{dS}{\mu(z)},$$

where $\mu(z)$ is the Levi form associated with the signed Euclidean distance to the boundary. When $m > 1$, it follows from the above proof that

$$\mathcal{N}(\lambda) \lesssim \lambda^{-m} \int\limits_{b\Omega} \bigl(\delta(z', 1/\sqrt{\lambda})\bigr)^{-1}\, dS(z'). \tag{6.16}$$

Heuristic arguments suggest that in this case the sharp asymptotic formula for $\mathcal{N}(\lambda)$ should involve logarithmic functions of λ.

7. Hearing a finite type property

We prove the following slight generalization of Theorem 1.3 in this section.

Theorem 7.1. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$. Let $\mathcal{N}_q(\lambda)$ be the number of eigenvalues of the $\bar\partial$-Neumann Laplacian on $(0,q)$-forms that are less than or equal to λ. Suppose that for some q, $1 \leqslant q \leqslant n-1$, there exists positive constants M and C such that $\mathcal{N}_q(\lambda) \leqslant C\lambda^M$. Then $b\Omega$ has finite D_{n-1}-type $\leqslant 8M$.*

Recall that the D_{n-1}-type of $b\Omega$ is the maximal order of contact of $(n-1)$-dimensional regular complex varieties with $b\Omega$ [18]. It was shown by D'Angelo [17] that the D_{n-1}-type is identical to the second entry in Catlin's multitype [7].

The following lemma is well known (see [25] for the two-dimensional case and [62] for the general case). Throughout this section, we will use z' to denote the first $(n-1)$-tuple of $z \in \mathbb{C}^n$.

Lemma 7.2. *Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$. Assume that the D_{n-1}-type of $b\Omega$ at z^0 is $\geqslant 2m$. Then there exists a neighborhood U of z^0 and a biholomorphic map $w = \Psi(z)$ from U into $\mathbb{C}^n$ such that $\Psi(z^0) = 0$ and*

$$\Psi(\Omega \cap U) = \bigl\{w \in \mathbb{C}^n \bigm| |w'| < 1,\ |\operatorname{Im} w_n| < 1, \rho(w) = \operatorname{Re} w_n + h(w', \operatorname{Im} w_n) < 0\bigr\},$$

where $h(w', \operatorname{Im} w_n) = f(w') + (\operatorname{Im} w_n)\cdot g(w') + \sigma(w', \operatorname{Im} w_n)$ with $|f(w')| \lesssim |w'|^{2m}$, $|g(w')| \lesssim |w'|^{m+1}$, and $|\sigma(w', \operatorname{Im} w_n)| \lesssim (\operatorname{Im} w_n)^2$.

We now prove Theorem 7.1. Assume that the D_{n-1}-type of $b\Omega$ at z^0 is $\geqslant 2m$. We apply Lemma 7.2 and keep its notations.

Write $w_n = s + it$. Let $b(t)$ be the cut-off function constructed in the paragraph preceding Lemma 2.3. We first extend $b(t)$ to the whole complex plane as follows. Let χ be any smooth cut-off function supported on $(-2, 2)$ and identically 1 on $(-1, 1)$ and let

$$B(w_n) = \bigl(b(t) - ib'(t)s - b''(t)s^2/2\bigr)\chi\bigl(s/\bigl(1 + |t|^2\bigr)\bigr).$$

Then $B(0,t)=b(t)$ and $|\partial B(w_n)/\partial \bar{w}_n| \lesssim |s|^2$.

Let

$$L_j = \frac{\partial\rho}{\partial w_n}\frac{\partial}{\partial w_j} - \frac{\partial\rho}{\partial w_j}\frac{\partial}{\partial w_n}, \quad 1 \leqslant j \leqslant n-1, \quad \text{and} \quad L_n = \sum_{j=1}^{n} \frac{\partial\rho}{\partial \bar{w}_j}\frac{\partial}{\partial w_j}.$$

Let

$$\tilde{L}_j = \left(\Psi^{-1}\right)_*(L_j), \quad 1 \leqslant j \leqslant n-1, \quad \text{and} \quad \tilde{L}_n = \sum_{j=1}^{n} \frac{\partial\tilde{\rho}}{\partial \bar{z}_j}\frac{\partial}{\partial z_j},$$

where $\tilde{\rho}(z) = \rho\circ\Psi(z)$. Then $\tilde{L}_j$, $1 \leqslant j \leqslant n$, form a basis for $T^{1,0}(\mathbb{C}^n)$ in a neighborhood of z^0. Replacing $\tilde{L}_j$ by the product of $\tilde{L}_j$ with an appropriate cut-off function, we may assume that $\tilde{L}_j$ is supported in U. Let $\hat{L}_j$, $1 \leqslant j \leqslant n$, be the vector fields obtained after performing the Gram–Schmidt process on $\tilde{L}_j$. Let $\hat{\omega}_j$, $1 \leqslant j \leqslant n$, be the dual basis of $\hat{L}_j$. Let $a(w')$ be a smooth function identically 1 near the origin and compactly supported in the unit ball in $\mathbb{C}^{n-1}$. For any positive integers j and for any positive integer k such that $2^{mj-1}/j \leqslant k \leqslant 2^{mj}/j$, let

$$f_{j,k}(w) = \sqrt{k}4^{(m+n-1)j}a\left(4^j w'\right)B\left(4^{mj}w_n\right)e^{2\pi k 4^{mj} w_n}.$$

Let $g_{j,k}(z) = f_{j,k}(\Psi(z))\cdot(J\Psi(z))$. Let

$$u_{j,k} = g_{j,k}(z)\bar{\hat{\omega}}_1 \wedge \cdots \wedge \bar{\hat{\omega}}_{n-1}.$$

Then for any sufficiently large j, $u_{j,k}$ is a compactly supported smooth $(0,n-1)$-form in $\mathrm{Dom}(Q_{n-1})$. Moreover,

$$\begin{aligned}\|u_{j,k}\|_\Omega^2 &= \int\limits_{\mathbb{C}^{n-1}} dV(w') \int\limits_{\mathbb{R}} dt \int\limits_{-\infty}^{-h(w',t)} \left|f_{j,k}(w)\right|^2 ds \\ &= \sqrt{k} \int\limits_{\mathbb{C}^{n-1}} \left|a(\tilde{w}')\right|^2 dV(\tilde{w}') \int\limits_{\mathbb{R}} d\tilde{t} \int\limits_{-\infty}^{-4^{mj}h(4^{-j}\tilde{w}',4^{-mj}\tilde{t})} \left|B(\tilde{s},\tilde{t})\right|^2 e^{4\pi k\tilde{s}}\, d\tilde{s}.\end{aligned}$$

(After the substitutions $\tilde{w}' = 4^j w'$, $\tilde{w}_n = 4^{mj}w_n$.) Since $|4^{mj}h(4^{-j}\tilde{w}',4^{-mj}\tilde{t})| \lesssim 4^{-mj}$ and $k4^{-mj} \leqslant j^{-2}2^{-mj}$, we have

$$\begin{aligned}\|u_{j,k}\|_\Omega^2 &\lesssim \sqrt{k} \int\limits_{\mathbb{C}^{n-1}} \left|a(\tilde{w}')\right|^2 dV(\tilde{w}') \int\limits_{\mathbb{R}} d\tilde{t} \int\limits_{-\infty}^{C4^{-mj}} \left|B(\tilde{s},\tilde{t})\right|^2 e^{4\pi k\tilde{s}}\, d\tilde{s} \\ &\lesssim \int\limits_{\mathbb{C}^{n-1}} \left|a(\tilde{w}')\right|^2 dV(\tilde{w}') \int\limits_{|\tilde{t}|<1} e^{Ck4^{-mj}}\, d\tilde{t} \lesssim 1.\end{aligned}$$

Similarly, $\|u_{j,k}\|^2 \gtrsim 1$. Therefore, $\|u_{j,k}\|_\Omega^2 \approx 1$. Furthermore, after a substitution as above, we have that for any k, l such that $2^{mj-1}/j \leqslant k, l \leqslant 2^{mj}/j$, $\langle u_{j,k}, u_{j,l}\rangle$ equals

$$\sqrt{kl}\int_{\mathbb{C}^n}\left|a(\tilde{w}')\right|^2 dV(\tilde{w}')\int_{\mathbb{R}} d\tilde{t}\int_{-\infty}^{-4^{mj}h(4^{-j}\tilde{w}',4^{-mj}\tilde{t})}\left|B(\tilde{s},\tilde{t})\right|^2 e^{2\pi((k+l)\tilde{s}+i(k-l)\tilde{t})}\,d\tilde{s}.$$

Let A be the above expression with the upper limit in the last integral replaced by 0 and let B likewise be the above expression with the lower limit of the last integral replaced by 0. Thus $\langle u_{j,k}, u_{j,l}\rangle = A + B$. It is easy to see that

$$\begin{aligned}|B| &\leqslant \sqrt{kl}\int_{\mathbb{C}^{n-1}}\left|a(\tilde{w}')\right|^2 dV(\tilde{w}')\int_{\mathbb{R}} d\tilde{t}\int_{0}^{-4^{mj}h(4^{-j}\tilde{w}',4^{-mj}\tilde{t})}\left|B(\tilde{s},\tilde{t})\right|^2 e^{2\pi(k+l)\tilde{s}}\,d\tilde{s}\\ &\lesssim \frac{\sqrt{kl}}{k+l}\left(e^{C(k+l)4^{-mj}}-1\right)\lesssim (k+l)4^{-mj}\lesssim j^{-1}2^{-mj}.\end{aligned}$$

To estimate $|A|$, we first observe that by Lemma 2.3, for $k \neq l$,

$$A = \sqrt{kl}\int_{\mathbb{C}^{n-1}}\left|a(\tilde{w}')\right|^2 dV(\tilde{w}')\int_{\mathbb{R}} d\tilde{t}\int_{-\infty}^{0}\left(\left|B(\tilde{s},\tilde{t})\right|^2-\left|B(0,\tilde{t})\right|^2\right)e^{2\pi((k+l)\tilde{s}+i(k-l)\tilde{t})}\,d\tilde{s}.$$

Hence

$$|A| \lesssim \sqrt{kl}\int_{|\tilde{w}'|<1}\left|a(\tilde{w}')\right|^2 dV(\tilde{w}')\int_{-1}^{1} d\tilde{t}\int_{-\infty}^{0}\tilde{s}e^{2\pi(k+l)\tilde{s}}\,d\tilde{s}\lesssim \sqrt{kl}/(k+l)^2.$$

Therefore, for sufficiently large j and for any k, l such that $2^{mj-1}/j \leqslant k, l \leqslant 2^{mj}/j$, $k \neq l$, we have

$$\left|\langle u_{j,k}, u_{j,l}\rangle\right| \lesssim j^{-1}2^{-mj}.$$

For any k such that $2^{mj-1}/j \leqslant k \leqslant 2^{mj}/j$ and for any $c_k \in \mathbb{C}$, we have

$$\begin{aligned}\left\|\sum_k c_k u_{j,k}\right\|^2 &= \sum_k |c_k|^2\|u_{j,k}\|^2 - \sum_{\substack{k,l\\ k\neq l}} c_k\bar{c}_l\langle u_{j,k},\ u_{j,l}\rangle\\ &\geqslant \sum_k |c_k|^2\|u_{j,k}\|^2 - j^{-1}2^{-mj}\left(\sum_k |c_k|\right)^2\\ &\gtrsim \left(1-j^{-2}\right)\sum_k |c_k|^2 \gtrsim \sum_k |c_k|^2,\end{aligned}$$

where the summations are taken over all integers between $2^{mj-1}/j$ and $2^{mj}/j$.

Since each $\hat{L}_k$, $1 \leqslant k \leqslant n-1$, is a linear combination of $\tilde{L}_1, \dots, \tilde{L}_k$, and $\hat{L}_n$ is just the normalization of $\tilde{L}_n$, it follows that

$$\begin{aligned} Q_{n-1}(u_{j,k}, u_{j,k}) &\lesssim \|g_{j,k}\|^2_\Omega + \sum_{l=1}^{n-1}\big(\|\bar{\tilde{L}}_l g_{j,k}\|^2_\Omega + \|\tilde{L}_l g_{j,k}\|^2_\Omega\big) + \|\bar{\tilde{L}}_n g_{j,k}\|^2_\Omega \\ &\lesssim \|f_{j,k}\|^2_{\Psi(\Omega\cap U)} + \sum_{l=1}^{n-1}\|L_l f_{j,k}\|^2_{\Psi(\Omega\cap U)} + \|\bar{\partial} f_{j,k}\|^2_{\Psi(\Omega\cap U)}. \end{aligned}$$

For $1 \leqslant l \leqslant n-1$, $\|L_l f_{j,k}\|^2_{\Psi(\Omega\cap U)}$ is bounded from above by a constant multiple of

$$\begin{aligned} &\left\|\frac{\partial\rho}{\partial w_n}\frac{\partial f_{j,k}}{\partial w_l}\right\|^2_{\Psi(\Omega\cap U)} + \left\|\frac{\partial\rho}{\partial w_l}\frac{\partial f_{j,k}}{\partial w_n}\right\|^2_{\Psi(\Omega\cap U)} \\ &\quad\lesssim 4^{2j} + \int\limits_{|\tilde{w}'|<1} \big|a(\tilde{w}')\big|^2 dV(\tilde{w}') \int\limits_{-1}^{1} d\tilde{t} \int\limits_{-\infty}^{-4^{mj}h(4^{-j}\tilde{w}', 4^{-mj}\tilde{t})} \Big(\big(4^{mj}|\nabla B| \\ &\qquad + k4^{mj}|B|\big)\big(\big|4^{-j}\tilde{w}'\big|^{2m-1} + \big|4^{-mj}\tilde{t}\big|\cdot\big|4^{-j}\tilde{w}'\big|^{m-1} + \big|4^{-mj}\tilde{t}\big|^2\big)\Big)^2 k e^{4\pi k\tilde{s}}\, d\tilde{s} \\ &\quad\lesssim 4^{2j}. \end{aligned}$$

Furthermore, $\|\bar{\partial} f_{j,k}\|^2_{\Psi(\Omega\cap U)}$ is

$$\begin{aligned} &\lesssim \sum_{l=1}^{n}\left\|\frac{\partial f_{j,k}}{\partial \bar{w}_l}\right\|^2_{\Psi(\Omega\cap U)} \\ &\lesssim 4^{2j} + 4^{2mj}k \int\limits_{|\tilde{w}'|<1} \big|a(\tilde{w}')\big|^2 dV(\tilde{w}') \int\limits_{-1}^{1} d\tilde{t} \int\limits_{-\infty}^{-4^{mj}h(4^{-j}\tilde{w}', 4^{-mj}\tilde{t})} \left|\frac{\partial B}{\partial \bar{w}_n}\right|^2 e^{4\pi k\tilde{s}}\, d\tilde{s} \\ &\lesssim 4^{2j} + 4^{2mj}k \int\limits_{|\tilde{w}'|<1} \big|a(\tilde{w}')\big|^2 dV(\tilde{w}') \int\limits_{-1}^{1} dt \int\limits_{-\infty}^{-4^{mj}h(4^{-j}\tilde{w}', 4^{-mj}\tilde{t})} \tilde{s}^4 e^{4\pi k\tilde{s}}\, d\tilde{s} \\ &\lesssim 4^{2j} + 4^{2mj}k^{-4} \lesssim 4^{2j}. \end{aligned}$$

Therefore, we have

$$Q_{n-1}(u_{j,k}, u_{j,k}) \lesssim 4^{2j}.$$

We now invoke the hypothesis of Theorem 7.1. Since $\mathcal{N}_q(\lambda) \lesssim \lambda^M$, $\lambda_j(\Box_q) \gtrsim j^{1/M}$. It follows from Proposition 2.2 that $\lambda_j(\Box_{n-1}) \gtrsim j^{1/M}$. By Lemma 2.1, for all sufficiently large j, there exists an integer $k_0 \in [2^{mj-1}/j, 2^{mj}/j]$ such that

$$Q_{n-1}(u_{j,k_0}, u_{j,k_0}) \gtrsim \big(2^{mj}/j\big)^{1/M}.$$

Therefore, $4^{2j} \gtrsim (2^{mj}/j)^{1/M}$. Hence $2m \leqslant 8M$. We thus conclude the proof of Theorem 7.1.

Acknowledgments

The author is indebted to Professors M. Christ, H. Jacobowitz, J.J. Kohn, T. Ohsawa, N. Stanton, and Y.-T. Siu for stimulating conversations and kind encouragement.

References

[1] Richard Beals, Peter C. Greiner, Nancy K. Stanton, The heat equation on a CR manifold, J. Differential Geom. 20 (2) (1984) 343–387.
[2] Richard Beals, Nancy K. Stanton, The heat equation for the $\bar\partial$-Neumann problem. I, Comm. Partial Differential Equations 12 (4) (1987) 351–413.
[3] Richard Beals, Nancy K. Stanton, The heat equation for the $\bar\partial$-Neumann problem. II, Canad. J. Math. 40 (2) (1988) 502–512.
[4] Thomas Bloom, Ian Graham, On "type" conditions for generic real submanifolds of $\mathbb{C}^n$, Invent. Math. 40 (3) (1977) 217–243.
[5] Harold P. Boas, Emil J. Straube, Global regularity of the $\bar\partial$-Neumann problem: A survey of the L^2-Sobolev theory, in: M. Schneider, Y.-T. Siu (Eds.), Several Complex Variables, in: Math. Sci. Res. Inst. Publ., vol. 37, 1999, pp. 79–112.
[6] Peter Buser, John Conway, Peter Doyle, Klaus-Dieter Semmler, Some planar isospectral domains, Int. Math. Res. Not. 9 (1994) 391–400.
[7] David Catlin, Necessary conditions for subellipticity of the $\bar\partial$-Neumann problem, Ann. of Math. (2) 117 (1) (1983) 147–171.
[8] David Catlin, Boundary invariants of pseudoconvex domains, Ann. of Math. (2) 120 (3) (1984) 529–586.
[9] David Catlin, Global regularity of the $\bar\partial$-Neumann problem, in: Yum-Tong Siu (Ed.), Complex Analysis of Several Variables, in: Proc. Sympos. Pure Math., vol. 41, American Mathematical Society, 1984, pp. 39–49.
[10] David Catlin, Estimates of invariant metrics on pseudoconvex domains of dimension two, Math. Z. 200 (1989) 429–466.
[11] So-Chin Chen, Mei-Chi Shaw, Partial differential equations in several complex variables, AMS/IP, 2000.
[12] Michael Christ, Regularity properties of the $\bar\partial_b$ equation on weakly pseudoconvex CR manifolds of dimension 3, J. Amer. Math. Soc. 1 (3) (1988) 587–646.
[13] Michael Christ, Remarks on global irregularity in the $\bar\partial$-Neumann problem, in: M. Schneider, Y.-T. Siu (Eds.), Several Complex Variables, in: Math. Sci. Res. Inst. Publ., vol. 37, 1999, pp. 161–198.
[14] Michael Christ, private communications.
[15] Michael Christ, Siqi Fu, Compactness in the $\bar\partial$-Neumann problem, magnetic Schrödinger operators, and the Aharonov–Bohm effect, Adv. Math. 197 (2005) 1–40.
[16] John D'Angelo, Real hypersurfaces, orders of contact, and applications, Ann. of Math. (2) 115 (3) (1982) 615–637.
[17] John D'Angelo, Finite type conditions for real hypersurfaces in $\mathbb{C}^n$, in: S.G. Krantz (Ed.), Complex Analysis Seminar, University Park, 1986, in: Lecture Notes in Math., vol. 1268, Springer-Verlag, 1987, pp. 83–102.
[18] John D'Angelo, Several Complex Variables and the Geometry of Real Hypersurfaces, Stud. Adv. Math., CRC Press, Boca Raton, FL, 1993.
[19] John P. D'Angelo, J.J. Kohn, Subelliptic estimates and finite type, in: M. Schneider, Y.-T. Siu (Eds.), Several Complex Variables, in: Math Sci. Res. Inst. Publ., vol. 37, 1999, pp. 199–232.
[20] Ingrid Daubechies, Ten Lectures on Wavelets, CBS–NSF Reg. Conf. Appl. Math., vol. 61, SIAM, Philadelphia, PA, 1992.
[21] E.B. Davies, Spectral Theory and Differential Operators, Cambridge University Press, 1995.
[22] Charles Fefferman, The Bergman kernel and biholomorphic mappings of pseudoconvex domains, Invent. Math. 26 (1974) 1–65.
[23] Charles L. Fefferman, Joseph J. Kohn, Hölder estimates on domains of complex dimension two and on three-dimensional CR manifolds, Adv. Math. 69 (2) (1988) 223–303.
[24] G.B. Folland, J.J. Kohn, The Neumann Problem for the Cauchy–Riemann Complex, Ann. of Math. Stud., vol. 75, Princeton University Press, 1972.
[25] J. Fornæss, N. Sibony, Construction of p.s.h. functions on weakly pseudoconvex domains, Duke Math. J. 58 (1989) 633–655.
[26] Siqi Fu, Hearing pseudoconvexity with the Kohn Laplacian, Math. Ann. 331 (2005) 475–485.

[27] Siqi Fu, Emil J. Straube, Compactness of the $\bar{\partial}$-Neumann problem on convex domains, J. Funct. Anal. 159 (1998) 629–641.
[28] Siqi Fu, Emil J. Straube, Compactness in the $\bar{\partial}$-Neumann problem, in: Complex Analysis and Geometry, Proceedings of Ohio State University Conference, vol. 9, Walter De Gruyter, 2001, pp. 141–160.
[29] Siqi Fu, Emil J. Straube, Semi-classical analysis of Schrödinger operators and compactness in the $\bar{\partial}$-Neumann problem, J. Math. Anal. Appl. 271 (2002) 267–282; Correction: J. Math. Anal. Appl. 280 (2003) 195–196.
[30] Lars Gårding, On the asymptotic distribution of the eigenvalues and eigenfunctions of elliptic differential operators, Math. Scand. 1 (1953) 237–255.
[31] C. Gordon, D. Webb, S. Wolpert, One cannot hear the shape of a drum, Bull. Amer. Math. Soc. (N.S.) 27 (1992) 134–138.
[32] E. Hernández, G. Weiss, A First Course on Wavelets, CRC Press, 1996.
[33] Lars Hörmander, L^2 estimates and existence theorems for the $\bar{\partial}$ operator, Acta Math. 113 (1965) 89–152.
[34] Lars Hörmander, An Introduction to Complex Analysis in Several Variables, third ed., North-Holland, 1990.
[35] M. Kac, Can one hear the shape of a drum?, Amer. Math. Monthly 73 (1966) 1–23.
[36] J.J. Kohn, Harmonic integrals on strongly pseudo-convex manifolds, I, Ann. of Math. (2) 78 (1963) 112–148.
[37] J.J. Kohn, Harmonic integrals on strongly pseudo-convex manifolds, II, Ann. of Math. (2) 79 (1964) 450–472.
[38] J.J. Kohn, Boundary behavior of $\bar{\partial}$ on weakly pseudo-convex manifolds of dimension two, J. Differential Geom. 6 (1972) 523–542.
[39] J.J. Kohn, Subellipticity of the $\bar{\partial}$-Neumann problem on pseudo-convex domains: Sufficient conditions, Acta Math. 142 (1979) 79–122.
[40] J.J. Kohn, L. Nirenberg, Non-coercive boundary value problems, Comm. Pure Appl. Math. 18 (1965) 443–492.
[41] Steven G. Krantz, Partial Differential Equations and Complex Analysis, CRC Press, Boca Raton, FL, 1992.
[42] Steven G. Krantz, Function Theory of Several Complex Variables, second ed., AMS Chelsea Publishing, Providence, RI, 2001.
[43] H. Laufer, On sheaf cohomology and envelopes of holomorphy, Ann. of Math. 84 (1966) 102–118.
[44] P.G. Lemarié, Y. Meyer, Ondelettes et bases hilbertiannes, Rev. Mat. Iberoamericana 2 (1986) 1–18.
[45] P. Matheos, A Hartogs domain with no analytic discs in the boundary for which the $\bar{\partial}$-Neumann problem is not compact, UCLA PhD thesis, 1997.
[46] Jeffery D. McNeal, Boundary behavior of the Bergman kernel function in C^2, Duke Math. J. 58 (2) (1989) 499–512.
[47] Guy Metivier, Spectral asymptotics for the $\bar{\partial}$-Neumann problem, Duke Math. J. 48 (1981) 779–806.
[48] A. Nagel, J.P. Rosay, E.M. Stein, S. Wainger, Estimates for the Bergman and Szegö kernels in $\mathbb{C}^2$, Ann. of Math. 129 (1989) 113–149.
[49] Alexander Nagel, Elias M. Stein, The $\Box_b$-heat equation on pseudoconvex manifolds of finite type in $\mathbb{C}^2$, Math. Z. 238 (1) (2001) 37–88.
[50] Alexander Nagel, Elias M. Stein, Stephen Wainger, Balls and metrics defined by vector fields. I. Basic properties, Acta Math. 155 (1–2) (1985) 103–147.
[51] Takeo Ohsawa, Complete Kähler manifolds and function theory of several complex variables, Sugaku Expositions 1 (1988) 75–93.
[52] Takeo Ohsawa, Analysis of Several Complex Variables, Transl. Math. Monogr., vol. 211, American Mathematical Society, Providence, RI, 2002.
[53] Walter Rudin, Real and Complex Analysis, third ed., McGraw–Hill Book Co., New York, 1987.
[54] J.-P. Serre, Quelques problèmes globaux relatifs aux variétés de Stein, in: Colloque sur les Fonctions de Plusieurs Variables, Brussels, 1953, pp. 57–68.
[55] Nessim Sibony, Une classe de domaines pseudoconvexes, Duke Math. J. 55 (2) (1987) 299–319.
[56] E.J. Straube, Plurisubharmonic functions and subellipticity of the $\bar{\partial}$-Neumann problem on non-smooth domains, Math. Res. Lett. 4 (1997) 459–467.
[57] Nancy K. Stanton, The heat equation in several complex variables, Bull. Amer. Math. Soc. (N.S.) 11 (1) (1984) 65–84.
[58] Nancy K. Stanton, David S. Tartakoff, The heat equation for the $\bar{\partial}_b$-Laplacian, Comm. Partial Differential Equations 9 (7) (1984) 597–686.
[59] E.M. Stein, G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces, Princeton Univ. Press, Princeton, NJ, 1971.
[60] François Trèves, Basic Linear Partial Differential Equations, Pure and Appl. Math., vol. 62, Academic Press, New York–London, 1975.
[61] Joachim Weidmann, Linear Operators in Hilbert Spaces, Grad. Texts in Math., vol. 68, Springer-Verlag, 1980.
[62] Jiye Yu, Weighted boundary limits of the generalized Kobayashi–Royden metrics on weakly pseudoconvex domains, Trans. Amer. Math. Soc. 347 (1995) 587–614.